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How Much Knowledge is in a Knowledge Base? Introducing Knowledge Measures (Preliminary Report)

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Abstract. In this work we address the following question: can we measure how much knowledge a knowledge base represents?

We answer to this question (i) by describing properties (axioms) that a *knowledge measure* we believe should have in measuring the amount of knowledge of a knowledge base (kb); and (ii) provide a concrete example of such a measure, based on the notion of entropy.

We also introduce related kb notions such as (i) *accuracy*; (ii) *conciseness*; and (iii) *Pareto optimality*. Informally, they address the following questions: (i) how precise is a kb in describing the actual world? (ii) how succinct is a kb w.r.t. the knowledge it represents? and (iii) can we increase accuracy without decreasing conciseness, or vice-versa?

1 INTRODUCTION

A *knowledge base* (kb) is the main ingredient of a *knowledge-based system* (kbs), whose aim is to use its kb to reason and to solve complex problems within a specific application domain. A kb essentially represents facts about such a specific world by relying usually on a formal logic.

An extensive work has been carried out by typically addressing properties, computational complexity, inference systems and implementation optimisations of propositional logic, First-Order Logic, modal logics, epistemic logics, temporal logics, non-monotone logics, various logic programming frameworks, probabilistic/possibilistic logics (see, e.g. [4]), and many/multiple-valued logics and fuzzy logics (see, e.g. [7, 8]), to mention a few.

In this paper, we address a novel problem[2]: namely, can we measure how much knowledge a kb (and, thus, a kbs) contains? We answer to this question in the context of propositional logic by introducing the notion of *knowledge measure*, called κ-index. Namely, we define axioms that a κ-index we believe should have. Then, we introduce related notions such as *accuracy*, *conciseness*, and *Pareto optimality*: the first one defines how precise a kb is in describing the actual world, the second one defines how succinct a kb is w.r.t. the knowledge it represents, while the last one establishes when we may not increase accuracy without decreasing conciseness (or vice-versa). Eventually, we provided a concrete example of such measures, based on the notion of entropy.

Example 1. *An example in which k-indexes may possibly find their usefulness is belief revision (see, e.g. [5]). Belief revision is the process of changing beliefs to take into account a new piece of knowledge, which possibly may be in contradiction with the current belief base. Of course, several different ways for performing this operation may be possible. The notion of k-index may be used here e.g. to define a revision operator maximising the k-index of the revised believe base.*

Other cases can be found similarly (use the k-index when we have to make a choice among alternative knowledge bases) such as e.g. non-monotonic reasoning (see e.g. [3]). Indeed, in most non-monotonic reasoning frameworks typically one has to deal with multiple extensions*, i.e. consistent and incomparable sets of formulae that are the outcome of the underlying non-monotonic framework. It is common to consider in such cases e.g.* brave reasoning *(a query formula is entailed by some extension) or* sceptical reasoning *(a query formula is entailed by all extensions). Likewise belief revision, we may employ here the notion of knowledge measure to consider e.g. entailment according to minimal/maximal k-index (a query formula is entailed by some/all knowledge minimal/maximal extensions).*

In the following, we proceed as follows. The next section introduces preliminary notions we will rely on. Section 3 is the main part of this work in which we define knowledge measures, accuracy, conciseness and Pareto optimality, together with some properties of them. In Section 4 we provide a concrete example of κ-index, while in Section 5 we recap our contribution and illustrate topics for future work.

2 PRELIMINARIES

Let us introduce main notions about propositional logic we will rely on. Specifically, let Σ be a *finite*, non-empty alphabet of *propositional letters* (denoted $a, b, c, \ldots$)[3], where Σ is a subset of an infinite, denumerable alphabet Σ_U (the *universal* alphabet).

A *formula* (denoted ϕ, ψ), is built in the usual way from the connectives $\neg, \wedge, \vee, \rightarrow$ and $\leftrightarrow$ and the alphabet Σ. The set of formulae is denoted $\mathcal{L}_\Sigma$. A *literal* (denoted L) is either a propositional letter or its negation. For a literal L, *(i)* if $L = p$ (resp. if $L = \neg p$) then L is called a *positive* (resp. *negative*) literal; and *(ii)* with $\bar{L}$ we denote $\neg p$ (resp. p) if $L = p$ (resp. if $L = \neg p$). Let $\bar{\Sigma} = \{\neg p \mid p \in \Sigma\}$ and let $\Sigma^\star = \Sigma \cup \bar{\Sigma}$. Note that $\bar{\Sigma} = \{\bar{p} \mid p \in \Sigma\}$.

A *knowledge base* (kb) $\mathcal{K} = \{\phi_1, \ldots, \phi_n\}$ is a finite set of formulae ϕ_i. Given a kb $\mathcal{K}$, with $\bigwedge \mathcal{K}$ we denote the formula $\bigwedge_{\phi \in \mathcal{K}} \phi$. In the following, whenever we write $\mathcal{K}$, we consider $\bigwedge \mathcal{K}$ instead, unless stated otherwise. Given a formula ϕ, with $\Sigma_\phi \subseteq \Sigma$ we denote the set of propositional letters occurring in ϕ. We define the *length* of ϕ,

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[2] To the best of our knowledge.

[3] All symbols we are going to use may have an optional sup- or sub-script.

denoted $|\phi|$, inductively as usual: for $p \in \Sigma$, $|p| = 1$, $\neg\phi = 1 + |\phi|$, $|\phi \vee \psi| = |\phi \wedge \psi| = |\phi \to \psi| = |\phi \leftrightarrow \psi| = 1 + |\phi| + |\psi|$.

An *interpretation* $\mathcal{I}$ w.r.t. Σ[4] is a set of literals such that all propositional letters in Σ occur exactly once in $\mathcal{I}$. With $\mathbb{I}_\Sigma$, or simply $\mathbb{I}$, we denote the set of all interpretations w.r.t. Σ. For $p \in \Sigma$, if $p \in \mathcal{I}$ (resp. $\neg p \in \mathcal{I}$) then we say that p is *true*, or also *positive* (resp. *false*, or also *negative*) in $\mathcal{I}$. We may also denote an interpretation $\mathcal{I}$ as the concatenation of the literals occurring in $\mathcal{I}$ with the convention to replace a negative literal $\neg p$ with $\bar{p}$ (e.g. the interpretation $\{p, \neg q\}$ may be denoted as well as $p\bar{q}$).

The notion of $\mathcal{I}$ *is a model of* (*satisfies*) a formula, denoted $\mathcal{I} \models \phi$, is then defined inductively as usual: $\mathcal{I} \models L$ iff $L \in \mathcal{I}$, $\mathcal{I} \models \neg\phi$ iff $\mathcal{I} \not\models \phi$, $\mathcal{I} \models \phi \wedge \psi$ iff $\mathcal{I} \models \phi$ and $\mathcal{I} \models \psi$, $\mathcal{I} \models \phi \vee \psi$ iff $\mathcal{I} \models \phi$ or $\mathcal{I} \models \psi$, $\mathcal{I} \models \phi \to \psi$ iff $\mathcal{I} \models \neg\phi \vee \psi$, and eventually $\mathcal{I} \models \phi \leftrightarrow \psi$ iff $\mathcal{I} \models (\phi \to \psi) \wedge (\psi \to \phi)$. Furthermore, we say that ϕ is *satisfiable* (resp. *unsatisfiable*) if it has (resp. has no) model.

We will also use two special symbols: with $\bot$ we will denote the formula that, by definition, has no models, while with $\top$ we denote the formula that, by definition, is satisfied in all interpretations. We impose that $\top$ (resp. $\bot$) cannot occur in any other formula different than $\top$ (resp. $\bot$) itself[5].

Given a formula ϕ, we define $\mathcal{M}(\phi, \Sigma)$ as the set of models of ϕ w.r.t. Σ. We may also write $\mathcal{M}(\phi)$ in place of $\mathcal{M}(\phi, \Sigma_\phi)$.

With $F_{\mathcal{M}(\phi,\Sigma)}$ we denote the formula (in *Disjunctive Normal Form* -DNF)

$$F_{\mathcal{M}(\phi,\Sigma)} = \bigvee_{\mathcal{I} \in \mathcal{M}(\phi,\Sigma)} \big(\bigwedge_{L \in \mathcal{I}} L\big)$$

with the condition that $F_{\mathcal{M}(\phi,\Sigma)} = \bot$ if $\mathcal{M}(\phi, \Sigma) = \emptyset$ and, thus, $F_{\mathcal{M}(\bot,\Sigma)} = \bot$. Moreover, we define $F_{\mathcal{M}(\top,\Sigma)} = \top$.

We say that ϕ *entails* ψ, denoted $\phi \models \psi$, or also $\phi \models^{\leq} \psi$, if $\mathcal{M}(\phi, \Sigma) \subseteq \mathcal{M}(\psi, \Sigma)$. We write $\phi \models^{<} \psi$ if $\phi \models \psi$ and $\psi \not\models \phi$. We say that ϕ and ψ are *equivalent*, denoted $\phi \equiv \psi$, if $\phi \models \psi$ and $\psi \models \phi$, *i.e.* $\models \phi \leftrightarrow \psi$. Note that, ϕ, $F_{\mathcal{M}(\phi,\Sigma)}$ and $F_{\mathcal{M}(\phi,\Sigma_\phi)}$ are pairwise equivalent.

We define a *substitution* θ as a set

$$\theta = \{p_1/L_1, \ldots, p_{|\Sigma|}/L_{|\Sigma|}\}$$

such that each propositional letter in Σ occurs exactly once in $\{p_1, \ldots p_{|\Sigma|}\}$ as well as in $\{L_1, \ldots L_{|\Sigma|}\}$. That is, the function $\theta(p_i) = L_i$ is a bijection $\theta : \Sigma \to \Sigma^\star$. The identity substitution ϵ is defined as $\epsilon = \{p_1/p_1, \ldots p_{|\Sigma|}/p_{|\Sigma|}\}$. The intuition is that propositional letters may be renamed by literals. Specifically, given a substitution $\theta = \{p_1/L_1, \ldots p_{|\Sigma|}/L_{|\Sigma|}\}$, with (i) $\phi\theta$ we indicate the formula obtained from ϕ by replacing every occurrence of p_i in ϕ with L_i; (ii) with $\mathcal{I}\theta$ we indicate the interpretation obtained from $\mathcal{I}$ by replacing every occurrence of p_i in $\mathcal{I}$ with L_i (with the convention that double negations are normalised[6]); and (iii) for a set of interpretations $\mathcal{M}$, with $\mathcal{M}\theta$ we denote the set

$$\mathcal{M}\theta = \{\mathcal{I}\theta \mid \mathcal{I} \in \mathcal{M}\} .$$

If $\mathcal{M}_1$ and $\mathcal{M}_2$ are two sets of interpretations, we write $\mathcal{M}_1 \leq M_2$ if there is a substitution θ such that $\mathcal{M}_1\theta \subseteq \mathcal{M}_2$. If the subset relation is strict, we write $\mathcal{M}_1 < \mathcal{M}_2$. We write $\mathcal{M}_1 \approx \mathcal{M}_2$ if both $\mathcal{M}_1 \leq \mathcal{M}_2$ and $\mathcal{M}_2 \leq \mathcal{M}_1$ hold. Furthermore, we say that a formula ϕ *k-entails* a formula ψ, denoted $\phi \models\!\!\!\approx \psi$ or also $\phi \models\!\!\!\approx^{\leq} \psi$, if $\mathcal{M}(\phi, \Sigma) \leq \mathcal{M}(\psi, \Sigma)$. We write $\phi \models\!\!\!\approx^{<} \psi$ if $\phi \models\!\!\!\approx \psi$ and $\psi \not\models\!\!\!\approx \phi$. Eventually, we say ϕ and ψ are *k-equivalent*, denoted $\phi \approx \psi$, if $\phi \models\!\!\!\approx \psi$ and $\psi \models\!\!\!\approx \phi$.

The following obvious proposition holds:

Proposition 2. *If $\phi \models \psi$ then $\phi \models\!\!\!\approx \psi$.*

Proof. Assume $\phi \models \psi$. Then $\mathcal{M}(\phi, \Sigma) \subseteq \mathcal{M}(\psi, \Sigma)$ and, thus, $\mathcal{M}(\phi, \Sigma)\epsilon \subseteq \mathcal{M}(\psi, \Sigma)$ holds. Therefore, $\phi \models\!\!\!\approx \psi$, which concludes. □

The converse is not true.

Example 3. *It is easily verified that using substitution $\theta = \{p/\neg q, q/p\}$, $p \models\!\!\!\approx \neg q$, but $p \not\models \neg q$.*

Corollary 4. *If $\phi \equiv \psi$ then $\phi \approx \psi$.*

In the following we will use the metavariable $\lhd$ to denote either $\leq$ or $<$ (*i.e.*, $\lhd \in \{\leq, <\}$).

Proposition 5. *If $\phi \models\!\!\!\approx^{\lhd} \psi$ then $|\mathcal{M}(\phi, \Sigma)| \lhd |\mathcal{M}(\psi, \Sigma)|$.*

Proof. Assume $\phi \models\!\!\!\approx \psi$. Then there is a substitution θ such that $\mathcal{M}(\phi, \Sigma)\theta \subseteq \mathcal{M}(\psi, \Sigma)$ and, thus, $|\mathcal{M}(\psi, \Sigma)| \geq |\mathcal{M}(\phi, \Sigma)\theta| = |\mathcal{M}(\phi, \Sigma)|$. The other case is shown similarly. □

Corollary 6. *If $\phi \equiv \psi$, or more generally, if $\phi \approx \psi$ then $|\mathcal{M}(\phi, \Sigma)| = |\mathcal{M}(\psi, \Sigma)|$.*

Given an interpretation $\mathcal{J}$ w.r.t. alphabet Σ and an interpretation $\mathcal{I}$ w.r.t. alphabet $\Sigma' \subseteq \Sigma$, we say that $\mathcal{J}$ is an *extension* of $\mathcal{I}$ if $\mathcal{I} \subseteq \mathcal{J}$. If $\mathcal{J}$ is an extension of $\mathcal{I}$, then the *residual* interpretation $\mathcal{J}[\mathcal{I}]$ w.r.t. alphabet $\Sigma \setminus \Sigma'$ is defined as $\mathcal{J} \setminus \mathcal{I}$. Furthermore, for a set $\mathcal{M}_1$ of interpretations w.r.t. Σ and a set $\mathcal{M}_2$ of interpretations w.r.t. $\Sigma' \subseteq \Sigma$, we define the *residual* $\mathcal{M}_1[\mathcal{M}_2]$ as the set of residual interpretations:

$$\mathcal{M}_1[\mathcal{M}_2] = \{\mathcal{J}[\mathcal{I}] \mid \mathcal{J} \in \mathcal{M}_1, \mathcal{I} \in \mathcal{M}_2, \mathcal{J} \text{ is an extension of } \mathcal{I}\} .$$

Example 7. *Consider*

$$\begin{aligned} \mathcal{M}_2 &= \{a\bar{b}, \bar{a}b\} \ (\Sigma' = \{a, b\}) \\ \mathcal{M}_1 &= \{a\bar{b}\bar{c}, a\bar{b}c, ab\bar{c}, abc\} \ (\Sigma = \{a, b, c\}) . \end{aligned}$$

Then

$$\mathcal{M}_1[\mathcal{M}_2] = \{\bar{c}, c\} \ (\Sigma \setminus \Sigma' = \{c\}) .$$

The following can easily be shown:

Proposition 8. *Given a formula ϕ, consider $\Sigma_\phi \subseteq \Sigma' \subseteq \Sigma$, $\mathcal{M}_1 = \mathcal{M}(\phi, \Sigma')$ and $\mathcal{M}_2 = \mathcal{M}(\phi, \Sigma_\phi)$. Then $\mathcal{M}_1[\mathcal{M}_2]$ is the set of all interpretations over $\Sigma' \setminus \Sigma_\phi$, i.e. $\mathcal{M}_1[\mathcal{M}_2] = \mathbb{I}_{\Sigma' \setminus \Sigma_\phi}$ and, thus, the formula $F_{\mathcal{M}_1[\mathcal{M}_2]}$ is equivalent to $\top$.*

3 KNOWLEDGE MEASURES

In the following we address basic principles of a knowledge measure for propositional knowledge bases we believe should hold.

3.1 Axioms

To start with, there are some principles we rely on to define a knowledge measure.

1. We consider a formula ϕ as a formal specification of the actual world, which is one of the models of ϕ.

[4] We may omit the reference to Σ if no ambiguity arises.
[5] Obviously, we also define $|\top| = |\bot| = 1$.
[6] $\neg\neg p \mapsto p$.

2. The more ϕ entails the more information ϕ represents.
3. The more models a formula has, the more uncertain we are about which is the actual world, which in turn means the less we know about the actual world.

A *knowledge measure* (also κ-*index*) w.r.t. a finite alphabet $\Sigma \subseteq \Sigma_U$ is a function

$$\kappa : 2^{\mathbb{I}_\Sigma} \to [0, \infty] .$$

For a set of interpretations $\mathcal{M} \in 2^{\mathbb{I}_\Sigma}$, we may also write $\kappa(\mathcal{M}, \Sigma)$ in place of $\kappa(\mathcal{M})$ when we want to emphasise that the involved alphabet is Σ[7].

We require that a κ-index has to satisfy the following axioms:

Axiom T: $\kappa(\mathbb{I}_\Sigma) = 0$ and $\kappa(\emptyset) = \infty$;
Axiom E: **if** $\mathcal{M}_1 \lhd \mathcal{M}_2$ **then** $\kappa(\mathcal{M}_1) \rhd \kappa(\mathcal{M}_2)$.

We extend κ to a function

$$\kappa : \mathcal{L}_\Sigma \times 2^{\Sigma_U} \to [0, \infty]$$

over formulae as follows:

$$\kappa(\phi, \Sigma) = \kappa(\mathcal{M}(\phi, \Sigma)) .$$

In the following, we define $\kappa(\phi)$ as a shorthand for $\kappa(\phi, \Sigma_\phi)$.

By definition we get immediately the following properties:

Proposition 9. *The following hold:*

1. *if* $\models \phi$ *then* $\kappa(\phi, \Sigma) = 0$*;*
2. *if* ϕ *is unsatisfiable then* $\kappa(\phi, \Sigma) = \infty$*;*
3. *if* $\phi \models \psi$ *and, more generally, if* $\phi \not\approx^{\lhd} \psi$ *then* $\kappa(\phi, \Sigma) \rhd \kappa(\psi, \Sigma)$.

Using the proposition above, we may rewrite axioms (T) and (E) also as

Axiom T: $\kappa(\top, \Sigma) = 0$ and $\kappa(\bot, \Sigma) = \infty$;
Axiom E: **if** $\phi \not\approx^{\lhd} \psi$ **then** $\kappa(\phi, \Sigma) \rhd \kappa(\psi, \Sigma)$.

Let us shortly explain the above axioms. Concerning axiom (T), if $\models \phi$ then ϕ is satisfied in all interpretations, *i.e.* $\mathcal{M}(\phi, \Sigma)$ is the set of all possible interpretations over Σ. This scenario depicts the maximal uncertainty we may have about which is the actual world described by ϕ and, thus, ϕ represents the least possible amount of knowledge, *i.e.* $\kappa(\phi, \Sigma) = 0$. Concerning axiom (E), assume $\phi \models \psi$. Then ϕ has less models than ψ, which means that there is less uncertainty about which is the actual model for ϕ compared to ψ. Moreover, whatever is entailed by ψ is also entailed by ϕ. Combining the two facts together means that ϕ represents more information about what is the actual world than ψ. That is, ϕ is more specific than ψ in describing which/what could be the actual world. Let us consider now k-entailment. Consider e.g. $\phi := p$ and $\psi := q$. Of course, neither $\phi \models \psi$ nor $\psi \models \phi$ hold. But, does ϕ represent more knowledge than ψ, or vice-versa? As $\phi \approx \psi$, by axiom (E) we have that $\kappa(\phi, \Sigma) = \kappa(\psi, \Sigma)$ instead, *i.e.* p and q represent exactly the same amount of knowledge. Essentially, our assumption here is that a knowledge measure is insensitive to symbol names, *i.e.* a symbol p represents as much knowledge as another symbol q. Of course, one may change such an assumption and assign to each propositional letter an a priori *mass* of information, which may differ from letter to letter. However, we do not address this here yet and leave it for future work.

[7] Recall that Σ is implicit, given $\mathcal{M}$.

Remark 10. *Please note that a knowledge measure somewhat has a relationship to a so-called* fuzzy measure *(see, e.g. [10]). In fact, we recap that a fuzzy measure in our context may be defined as a function*

$$g : 2^{\mathbb{I}} \to [0, \infty]$$

with

1. $g(\emptyset) = 0$*;*
2. *if* $\mathcal{M}_1 \subseteq \mathcal{M}_2$ *then* $g(\mathcal{M}_1) \leq g(\mathcal{M}_2)$

and, thus, there are some 'reversed' commonalities.

Remark 11 (About tautology)**.** *Note that if* $\models \phi$ *then for every formula* ψ*, we have that* $\psi \models \phi$ *and, thus, by axiom* (E) $\kappa(\psi, \Sigma) \geq \kappa(\phi, \Sigma)$ *has to hold, i.e.* $\kappa(\phi, \Sigma)$ *has to be as small as possible, which motivates* $\kappa(\phi, \Sigma) = 0$.

Remark 12. *(About unsatisfiability) Analogously to Remark 11, if* ϕ *is unsatisfiable then for every formula* ψ*, we have that* $\phi \models \psi$ *and, thus, by axiom* (E) $\kappa(\phi, \Sigma) \geq \kappa(\psi, \Sigma)$ *has to hold. That is,* $\kappa(\phi, \Sigma)$ *has to be as large as possible, which motivates* $\kappa(\phi, \Sigma) = \infty$.

Note however, that axiom (E) does not imply axiom (T).

Example 13. *Consider* $\phi := p \wedge (p \to q)$ *and* $\psi := q$*. As* $\phi \models \psi$*, by Proposition 2 and axiom* (E)*, we have that* $\kappa(\phi, \Sigma) \geq \kappa(\psi, \Sigma)$.

Example 14. *Consider* $\phi := p \wedge (p \to q)$ *and* $\psi := r \wedge s$*. As* $\phi \not\approx \psi$*, then* $\kappa(\phi, \Sigma) \geq \kappa(\psi, \Sigma)$*. But, also* $\psi \not\approx \phi$ *holds*[8] *so we have* $\kappa(\psi, \Sigma) \geq \kappa(\phi, \Sigma)$ *and, thus,* $\kappa(\psi, \Sigma) = \kappa(\phi, \Sigma)$.

That is, the following holds:

Corollary 15. *If* $\phi \equiv \psi$*, or more generally, if* $\phi \approx \psi$ *then* $\kappa(\psi, \Sigma) = \kappa(\phi, \Sigma)$.

Proof. It suffices to apply axiom (E) to both $\phi \not\approx \psi$ and $\psi \not\approx \phi$. □

Corollary 16. *For all propositional letters* $p, q \in \Sigma$ *we have that* $\kappa(p, \Sigma) = \kappa(q, \Sigma)$.

Proof. Apply Proposition 15 to $p \approx q$. □

Next, there are two more axioms we would like a κ-index has to satisfy.

Axiom L: $\kappa(\mathcal{M}(\phi, \Sigma')) = \kappa(\mathcal{M}(\phi))$, for all $\Sigma_\phi \subseteq \Sigma' \subseteq \Sigma$;
Axiom M: **if** ϕ is satisfiable **then**

1. $0 \leq \kappa(\mathcal{M}(\phi)) \leq |\Sigma_\phi|$
2. **if** $|\mathcal{M}(\phi)| = 1$ **then** $\kappa(\mathcal{M}(\phi)) = |\Sigma_\phi|$.

Essentially, the rationale behind these axioms is the following: concerning axiom (L), this axiom says that, to what concerns κ-indexes, we may restrict our attention to Σ_ϕ, *i.e.* the set of all propositional letters occurring in a formula ϕ and, thus, symbols not occurring in ϕ do not contribute to represent additional knowledge[9] . Concerning axiom (M), this axiom tells us that a κ-index is bounded in the sense that a satisfiable formula may not represent more knowledge than the number of symbols it relies on and the bound is reached only if the formula exactly describes the actual world.

[8] In fact, $\phi \approx \psi$.
[9] Compare in contrast e.g. with the case of *Closed World Assumption* (CWA) [11].

As for axioms (T) and (E), we may rewrite the axioms (L) and (M) above as follows:

Axiom L: $\kappa(\phi, \Sigma') = \kappa(\phi)$, for all $\Sigma_\phi \subseteq \Sigma' \subseteq \Sigma$;
Axiom M: **if** ϕ is satisfiable **then**

1. $0 \leq \kappa(\phi) \leq |\Sigma_\phi|$
2. **if** $|\mathcal{M}(\phi)| = 1$ **then** $\kappa(\phi) = |\Sigma_\phi|$.

Eventually, we conclude with some properties of κ-indexes.

The following can also easily be shown:

Proposition 17. *Consider formulae ϕ and ψ. If $\phi \approx^{\lhd} \psi$ then $\kappa(\phi, \Sigma') \rhd \kappa(\psi, \Sigma')$ for all $\Sigma_\phi \cup \Sigma_\psi \subseteq \Sigma' \subseteq \Sigma$.*

Proof. By axiom (L) we have $\kappa(\phi, \Sigma') = \kappa(\phi) = \kappa(\phi, \Sigma)$ and $\kappa(\psi, \Sigma') = \kappa(\psi) = \kappa(\psi, \Sigma)$. By axiom (E), $\kappa(\phi, \Sigma') = \kappa(\phi) = \kappa(\phi, \Sigma) \rhd \kappa(\psi, \Sigma) = \kappa(\psi) = \kappa(\psi, \Sigma')$, which concludes. □

Proposition 18. *For sets of interpretations $\mathcal{M}_1, \mathcal{M}_2 \subseteq \mathbb{I}_\Sigma$ we have that:*

1. $\kappa(\mathcal{M}_1 \cap \mathcal{M}_2) \geq \max(\kappa(\mathcal{M}_1), \kappa(\mathcal{M}_2))$;
2. $\kappa(\mathcal{M}_1 \cup \mathcal{M}_2) \leq \min(\kappa(\mathcal{M}_1), \kappa(\mathcal{M}_2))$.

Proof. Concerning point 1: as $\mathcal{M}_1 \cap \mathcal{M}_2 \lhd \mathcal{M}_i$, it suffices to apply axiom (E). Concerning point 2: as $\mathcal{M}_1 \cup \mathcal{M}_2 \rhd \mathcal{M}_i$, again apply axiom (E). □

From the proposition above, we get immediately:

Corollary 19. *For formulae ϕ and ψ we have that:*

1. $\kappa(\phi \wedge \psi, \Sigma) \geq \max(\kappa(\phi, \Sigma), \kappa(\psi, \Sigma))$;
2. $\kappa(\phi \vee \psi, \Sigma) \leq \min(\kappa(\phi, \Sigma), \kappa(\psi, \Sigma))$.

We conclude the section with three additional measures related to a κ-index.

3.2 Accuracy

The accuracy indicates how precise a satisfiable formula is in describing the actual world. Specifically, an *accuracy measure*, or *α-index*, is a function

$$\alpha : \mathcal{L}_\Sigma \times 2^{\Sigma_U} \to [0, 1]$$

that has to satisfy the following axioms, where we use $\alpha(\phi)$ as a shorthand for $\alpha(\phi, \Sigma_\phi)$: for satisfiable formulae ϕ and ψ

Axiom A1: $\alpha(\phi, \Sigma') = \alpha(\phi)$, for $\Sigma_\phi \subseteq \Sigma' \subseteq \Sigma$;
Axiom A2: **if** $\kappa(\phi) \geq \kappa(\psi)$ **and** $|\Sigma_\phi| \leq |\Sigma_\psi|$ **then** $\alpha(\phi) \geq \alpha(\psi)$. The relation is strict if one of the two preconditions do so.

Essentially, axiom $(A1)$ says that the accuracy depends only on the symbols occurring in a formula, while axiom $(A2)$ aims at saying that if a formula represents more knowledge than another formula and uses also less symbols then it is more precise in describing the actual world.

The following propositions hold:

Proposition 20. *If $\phi \approx^{\lhd} \psi$ and $|\Sigma_\phi| \leq |\Sigma_\psi|$ then $\alpha(\phi) \geq \alpha(\psi)$.*

Proof. By Proposition 9 and axiom (L), $\kappa(\phi) \rhd \kappa(\psi)$ and, thus, by axiom $(A2)$, $\alpha(\phi) \geq \alpha(\psi)$ holds. □

Proposition 21. *Consider the function*

$$\bar{\alpha}(\phi) = \frac{\kappa(\phi)}{|\Sigma_\phi|} .$$

Then $\bar{\alpha}$ is an accuracy measure (and in particular, $0 \leq \bar{\alpha}(\phi) \leq 1$).

Proof. Immediate. □

3.3 Conciseness

The conciseness measure indicates how succinct a satisfiable formula is w.r.t. the knowledge it represents. Specifically, a *conciseness measure*, or *γ-index*, is a function

$$\gamma : \mathcal{L}_\Sigma \times 2^{\Sigma_U} \to [0, \infty]$$

that has to satisfy the following axioms, where we use $\gamma(\phi)$ as a shorthand for $\gamma(\phi, \Sigma_\phi)$: for satisfiable formulae ϕ and ψ

Axiom C1: $\gamma(\phi, \Sigma') = \gamma(\phi)$, for $\Sigma_\phi \subseteq \Sigma' \subseteq \Sigma$;
Axiom C2: **if** $\kappa(\phi) \geq \kappa(\psi)$ **and** $|\phi| \leq |\psi|$ **then** $\gamma(\phi) \geq \gamma(\psi)$. The relation is strict if one of the two preconditions do so.

Likewise accuracy, axiom $(C1)$ says that the conciseness depends only on the symbols occurring in a formula, while axiom $(C2)$ aims at saying that if a formula represents more knowledge than another formula and is also shorter then it is more concise in describing the actual world.

The following propositions hold:

Proposition 22. *If $\phi \approx^{\lhd} \psi$ and $|\phi| \leq |\psi|$ then $\gamma(\phi) \geq \gamma(\psi)$.*

Proof. By Proposition 9 and axiom (L), $\kappa(\phi) \rhd \kappa(\psi)$ and, thus, by axiom $(C2)$ $\gamma(\phi) \geq \gamma(\psi)$ holds. □

Proposition 23. *Consider the function*

$$\bar{\gamma}(\phi) = \frac{\kappa(\phi)}{|\phi|} .$$

Then $\bar{\gamma}$ is a conciseness measure and $0 \leq \bar{\gamma}(\phi) \leq \frac{|\Sigma_\phi|}{|\phi|} \leq 1$.

Proof. Immediate. □

Eventually, consider

$$0 \leq c_\phi = \frac{|\Sigma_\phi|}{|\phi|} \leq 1 . \quad (1)$$

We call c_ϕ the *conciseness factor* of ϕ. Now, it is easily verified that, using the conciseness factor, we may rewrite $\bar{\gamma}(\phi)$ as:

$$\bar{\gamma}(\phi) = c_\phi \cdot \bar{\alpha}(\phi) .$$

Of course, many alternatives for accuracy and conciseness may be worked out: however, we believe that accuracy (resp. conciseness) of a formula is monotone non-decreasing w.r.t. the formula's κ-index and non-increasing w.r.t. the formula's alphabet (resp. the formula's) size.

3.4 Pareto Optimality

The last notion we introduce is Pareto optimality of a satisfiable formula: that is, *for a given satisfiable formula, we cannot find another k-equivalent one by increasing accuracy without decreasing conciseness (and vice-versa).*

Example 24. *Consider ϕ and ψ as in Example 14. Then ϕ is not Pareto optimal as there is ψ with $\phi \approx \psi$ that has same accuracy, but is more concise (same amount of knowledge, but formula is shorter).*

Consider the function $\pi : \mathcal{L}_\Sigma \to 2^{\mathcal{L}_\Sigma}$ defined as[10]

$$\pi(\phi) \quad = \quad \arg \max_{\{\psi \mid \psi \approx \phi\}} \alpha(\psi) \cdot \gamma(\psi) \ .$$

The following can be shown:

Proposition 25. *For a satisfiable formula ϕ, $\phi \in \pi(\phi)$ iff ϕ is Pareto optimal.*

Proof. Assume $\phi \in \pi(\phi)$. Then, by definition, for all ψ such that $\psi \approx \phi$ we have $\alpha(\phi) \cdot \gamma(\phi) \geq \alpha(\psi) \cdot \gamma(\psi)$. This means that there cannot be a formula ψ with $\psi \approx \phi$ that increases at the same time accuracy and conciseness (as otherwise $\alpha(\phi) \cdot \gamma(\phi) < \alpha(\psi) \cdot \gamma(\psi)$ would hold). Therefore, ϕ is Pareto optimal. Vice-versa, assume ϕ is Pareto optimal. Therefore, if we cannot find another k-equivalent formula ψ by increasing accuracy without decreasing conciseness (or vice-versa), $\alpha(\phi) \cdot \gamma(\phi)$ is maximal and, thus, $\phi \in \pi(\phi)$. □

From Propositions 21, 23 and 25 it follows immediately that

Corollary 26. *For a satisfiable formula ϕ and function*

$$\bar{\pi}(\phi) = \arg \max_{\{\psi \mid \psi \approx \phi\}} \bar{\alpha}(\psi) \cdot \bar{\gamma}(\psi) \ ,$$

$\phi \in \bar{\pi}(\phi)$ iff ϕ is Pareto optimal.

In particular, we have

Corollary 27. *For a satisfiable formula ϕ, ϕ is Pareto optimal iff for all ψ such that $\psi \approx \phi$ we have that*

$$|\Sigma_\phi| \cdot |\phi| \leq |\Sigma_\psi| \cdot |\psi| \ .$$

Proof. By Corollary 26 ϕ is Pareto optimal iff $\phi \in \bar{\pi}(\phi)$, *i.e.* for all ψ such that $\psi \approx \phi$ and, thus, by Proposition 15 and axiom (L) $\kappa(\phi) = \kappa(\psi)$, we have $\bar{\alpha}(\phi) \cdot \bar{\gamma}(\phi) \geq \bar{\alpha}(\psi) \cdot \bar{\gamma}(\psi)$. Therefore, by Propositions 21 and 23, ϕ is Pareto optimal iff

$$\begin{aligned} \frac{\kappa(\phi)}{|\Sigma_\phi|} \cdot \frac{\kappa(\phi)}{|\phi|} \quad &\geq \quad \frac{\kappa(\psi)}{|\Sigma_\psi|} \cdot \frac{\kappa(\psi)}{|\psi|} \\ &= \quad \frac{\kappa(\phi)}{|\Sigma_\psi|} \cdot \frac{\kappa(\phi)}{|\psi|} \ , \end{aligned}$$

i.e., $|\Sigma_\phi| \cdot |\phi| \leq |\Sigma_\psi| \cdot |\psi|$, which concludes. □

Example 28 (Example 24 cont.). *It can be verified that ψ is Pareto optimal by Corollary 27. Indeed, $|\Sigma_\psi| = 2$, $|\psi| = 3$ and any other k-equivalent formula ψ' needs at least two symbols and, thus, $|\Sigma_{\psi'}| \geq 2$ and $|\psi| \geq 3$, i.e. $|\Sigma_\psi| \cdot |\psi| \leq |\Sigma_{\psi'}| \cdot |\psi'|$.*

We conclude by noting that we may use Corollary 27 to compute a Pareto optimal solution via a brute-force approach: that is, given a satisfiable formula ϕ, set $\phi_{po} := \phi$ and then enumerate all formulae ψ with $|\Sigma_\psi| \cdot |\psi| < |\Sigma_{\phi_{po}}| \cdot |\phi_{po}|$[11]. If such a ψ is also equivalent to ϕ then update ϕ_{po} with $\phi_{po} := \psi$. At the end, ϕ_{po} is a Pareto optimal solution w.r.t. ϕ.

[10] Recall that $\arg \max_x f(x) := \{x \mid \forall y. f(y) \leq f(x)\}$.
[11] In particular, either $|\Sigma_\psi|$ or $|\psi|$ is strictly smaller than $|\Sigma_{\phi_{po}}|$ or $|\phi_{po}|$, respectively.

4 AN ENTROPY BASED KNOWLEDGE MEASURE

We next provide a concrete example of a knowledge measure. We will restrict our attention to the more interesting case of satisfiable formulae only, unless specified otherwise.

The *entropy based κ-index* of a satisfiable formula ϕ, or simply *κ_h-index* of ϕ, denoted $\kappa_h(\phi, \Sigma)$, is defined as follows. At first, we define the *entropy of a formula* ϕ as the *entropy of a random variable* (see, e.g. [9]) ranging over models of ϕ. Specifically, for $\mathcal{I} \in \mathcal{M}(\phi, \Sigma)$, let

$$\wp(\mathcal{I}, \Sigma, \phi) = \frac{1}{|\mathcal{M}(\phi, \Sigma)|}$$

be the probability to randomly select a model of ϕ w.r.t. Σ.

Remark 29. *Note that in a more general setting we may also just envisage a probability distribution among the models. Also note that we do not consider a probability distribution among all possible interpretations over Σ (rather than models) as we focus on the models of a formula only and want to compute the entropy of generating one of these as the actual world a formula represents.*

Now, we define the *(Shannon) entropy* [9] of ϕ w.r.t. Σ as[12]

$$\mathcal{H}(\phi, \Sigma) \quad = \quad - \sum_{\mathcal{I} \in \mathcal{M}(\phi, \Sigma)} \wp(\mathcal{I}, \Sigma, \phi) \log_2 \wp(\mathcal{I}, \Sigma, \phi) \ .$$

Then, it is well known that under the uniform distribution

$$\mathcal{H}(\phi, \Sigma) \quad = \quad \log_2 |\mathcal{M}(\phi, \Sigma)|$$

holds. Note that $0 \leq \mathcal{H}(\phi, \Sigma) \leq |\Sigma|$. For convenience, we also define $\mathcal{H}(\phi) = \mathcal{H}(\phi, \Sigma_\phi)$.

Now, the *κ_h-index* of ϕ w.r.t. Σ is defined as

$$\kappa_h(\phi, \Sigma) = |\Sigma| - \mathcal{H}(\phi, \Sigma) \ ,$$

that is,

$$\kappa_h(\phi, \Sigma) = |\Sigma| - \log_2 |\mathcal{M}(\phi, \Sigma)| \ ,$$

and postulate $\kappa_h(\bot, \Sigma) = \infty$. We may write, as usual, $\kappa_h(\phi)$ in place of $\kappa_h(\phi, \Sigma_\phi)$ and, thus, have

$$\boxed{\kappa_h(\phi) = |\Sigma_\phi| - \log_2 |\mathcal{M}(\phi)|} \ . \tag{2}$$

We further extend κ_h to sets of interpretations as follows: for a set $\mathcal{M} \subseteq \mathbb{I}_\Sigma$ of interpretations w.r.t. Σ, we define

$$\kappa_h(\mathcal{M}) = \kappa_h(F_\mathcal{M}, \Sigma) \ . \tag{3}$$

The following proposition tells us that, concerning the κ_h-index, we may restrict our attention to Σ_ϕ, *i.e.* the set of all propositional letters occurring in a formula ϕ only.

Proposition 30. *Consider a formula ϕ and $\Sigma_\phi \subseteq \Sigma' \subseteq \Sigma$. Then*

$$\kappa_h(\phi, \Sigma') = \kappa_h(\phi) \ .$$

Proof. Let $n = |\Sigma' \setminus \Sigma_\phi|$. Then

$$\begin{aligned} |\Sigma'| \quad &= \quad |\Sigma_\phi| + n \\ |\mathcal{M}(\phi, \Sigma')| \quad &= \quad 2^n |\mathcal{M}(\phi, \Sigma_\phi)| \ . \end{aligned}$$

[12] We define by continuity $0 \log_2 0 = 0$.

Therefore, we have

$$\begin{aligned}\kappa_h(\phi,\Sigma') &= |\Sigma'| - \mathcal{H}(\phi,\Sigma') \\ &= |\Sigma_\phi| + n - \log_2|\mathcal{M}(\phi,\Sigma')| \\ &= |\Sigma_\phi| + n - \log_2 2^n|\mathcal{M}(\phi,\Sigma_\phi)| \\ &= |\Sigma_\phi| + n - n - \log_2|\mathcal{M}(\phi,\Sigma_\phi)| \\ &= |\Sigma_\phi| - \log_2|\mathcal{M}(\phi,\Sigma_\phi)| \\ &= \kappa_h(\phi,\Sigma_\phi) \\ &= \kappa_h(\phi)\ ,\end{aligned}$$

which concludes. □

An immediate consequence of Propositions 30 is that

Corollary 31. *κ_h satisfies axiom (L).*

Note that from Proposition 8 it also follows that

Proposition 32. *Given a formula ϕ, consider $\Sigma_\phi \subseteq \Sigma' \subseteq \Sigma$. Then $\kappa_h(\mathcal{M}(\phi,\Sigma')[\mathcal{M}(\phi)]) = 0$.*

Proof. Consider $\mathcal{M}_1 = \mathcal{M}(\phi,\Sigma')$ and $\mathcal{M}_2 = \mathcal{M}(\phi)$. By Proposition 8 we have that the formula $F_{M_1[\mathcal{M}_2]}$ is equivalent to $\top$. Therefore, by Eq. 3, $\kappa_h(M_1[\mathcal{M}_2]) = \kappa_h(\top,\Sigma'\setminus\Sigma_\phi) = |\Sigma'\setminus\Sigma_\phi| - \log_2 2^{|\Sigma'\setminus\Sigma_\phi|} = 0$, which concludes. □

Example 33. *Consider $\phi := p \wedge (p \rightarrow q)$. Then,*

$$\begin{aligned}\Sigma_\phi &= \{p,q\} \\ \mathcal{M}(\phi) &= \{pq\} \\ \wp(\mathcal{I},\Sigma_\phi,\phi) &= 1, \textit{ for all } \mathcal{I}\in\mathcal{M}(\phi) \\ \mathcal{H}(\phi) &= 0 \\ \kappa_h(\phi) &= 2\ .\end{aligned}$$

Example 34. *Consider $\phi := a \vee \neg a$. Then,*

$$\begin{aligned}\Sigma_\phi &= \{a\} \\ \mathcal{M}(\phi) &= \{a,\bar{a}\} \\ \wp(\mathcal{I},\Sigma_\phi,\phi) &= 0.5, \textit{ for all } \mathcal{I}\in\mathcal{M}(\phi) \\ \mathcal{H}(\phi) &= 1 \\ \kappa_h(\phi) &= 0\ .\end{aligned}$$

Remark 35. *Note that κ_h is monotone non-decreasing in the number of propositional letters that occur in a formula ϕ and monotone non-increasing w.r.t. the entropy ϕ. That is,* (i) *the more propositional letters occur in ϕ the higher may the knowledge it represents be; and,* (ii) *the more uncertain we are about which is the actual world ϕ is representing (i.e. the more models ϕ has), the less knowledge is represented by ϕ.*

We have that

Proposition 36. *If $\models \phi$ then $\kappa_h(\phi) = 0$ and, thus, κ_h satisfies axiom (T).*

Proof. Assume $\models \phi$. Then $|\mathcal{M}(\phi)| = 2^{|\Sigma_\phi|}$ and, thus, by Proposition 30, $\kappa_h(\phi,\Sigma) = \kappa_h(\phi) = 0$. □

By definition of κ_h, we also have that

$$0 \leq \kappa_h(\phi) \leq |\Sigma_\phi|$$

and, if $|\mathcal{M}(\phi)| = 1$ then $\kappa_h(\phi) = |\Sigma_\phi|$. Therefore,

Proposition 37. *κ_h satisfies axiom (M).*

Remark 38. *Note that, given an alphabet Σ, the formula*

$$\phi = \bigwedge_{p\in\Sigma} p$$

represents as much knowledge as possible, i.e.

$$\kappa_h(\phi) = |\Sigma| = |\Sigma_\phi|\ .$$

Let us now show that κ_h satisfies axiom (E) as well.

Proposition 39. *If $\phi \models^{\triangleleft}_{\approx} \psi$ then $\kappa_h(\phi) \rhd \kappa_h(\psi)$ and, thus, κ_h satisfies axiom (E).*

Proof. Assume $\phi \models^{\triangleleft}_{\approx} \psi$ holds. Then, by Proposition 5, $1 \leq |\mathcal{M}(\phi,\Sigma)| \lhd |\mathcal{M}(\psi,\Sigma)|$ holds. Now, by definition of κ_h and by Proposition 30 we have that

$$\begin{aligned}\kappa_h(\phi) &= \kappa_h(\phi,\Sigma) \\ &= |\Sigma| - \log_2|\mathcal{M}(\phi,\Sigma)| \\ &\rhd |\Sigma| - \log_2|\mathcal{M}(\psi,\Sigma)| \\ &= \kappa_h(\psi,\Sigma) \\ &= \kappa_h(\psi)\ .\end{aligned}$$

In particular, $\kappa(\phi,\Sigma) \rhd \kappa(\psi,\Sigma)$ and, thus κ_h satisfies axiom (E), which concludes. □

From Proposition 39 it follows immediately that

Corollary 40. *If $\phi \models \psi$ then $\kappa_h(\phi) \geq \kappa_h(\psi)$.*

Eventually, from Corollary 31, Propositions 36, 37 and 39, it follows that

Proposition 41. *The function κ_h is a knowledge measure.*

We conclude with the following property.

Proposition 42. *The κ_h-index is* additive *in the following sense: for formulae ϕ and ψ with $\Sigma_\phi \cap \Sigma_\psi = \emptyset$ we have that:*

$$\kappa_h(\phi\wedge\psi) = \kappa_h(\phi) + \kappa_h(\psi)\ .$$

Proof. Consider $\Sigma' = \Sigma_\phi \cup \Sigma_\psi = \Sigma_{\phi\wedge\psi}$. As $\Sigma_\phi \cap \Sigma_\psi = \emptyset$, it can be verified that $|\mathcal{M}(\phi\wedge\psi)| = |\mathcal{M}(\phi)|\cdot|\mathcal{M}(\psi)|$. Therefore,

$$\begin{aligned}\kappa_h(\phi\wedge\psi) &= |\Sigma_{\phi\wedge\psi}| - \log_2|\mathcal{M}(\phi\wedge\psi)| \\ &= |\Sigma_{\phi\wedge\psi}| - \log_2(|\mathcal{M}(\phi)||\mathcal{M}(\psi)|) \\ &= |\Sigma_{\phi\wedge\psi}| - \log_2|\mathcal{M}(\phi)| - \log_2|\mathcal{M}(\psi)| \\ &= |\Sigma_\phi| + |\Sigma_\psi| - \log_2|\mathcal{M}(\phi)| - \log_2|\mathcal{M}(\psi)| \\ &= \kappa_h(\phi) + \kappa_h(\psi)\ .\end{aligned}$$

□

Accuracy, Conciseness & Pareto optimality. Concerning accuracy, let us consider

$$\begin{aligned}\alpha_h(\phi,\Sigma) &= \frac{\kappa_h(\phi,\Sigma)}{|\Sigma_\phi|} \\ &= \frac{\kappa_h(\phi)}{|\Sigma_\phi|} \\ &= \alpha_h(\phi) \\ &= \frac{|\Sigma_\phi| - \log_2|\mathcal{M}(\phi)|}{|\Sigma_\phi|}\end{aligned}$$

So, with

$$\boxed{\alpha_h(\phi) = 1 - \frac{\log_2 |\mathcal{M}(\phi)|}{|\Sigma_\phi|}} \quad (4)$$

we denote the *entropy-based accuracy measure*, called α_h*-index*. Of course, by definition,

$$0 \leq \alpha_h(\phi) \leq 1 \ .$$

Concerning conciseness, consider

$$\begin{aligned} \gamma_h(\phi, \Sigma) &= \frac{\kappa_h(\phi, \Sigma)}{|\phi|} \\ &= \frac{\kappa_h(\phi)}{|\phi|} \\ &= \gamma_h(\phi) \ . \end{aligned}$$

So, with

$$\boxed{\gamma_h(\phi) = \frac{|\Sigma_\phi| - \log_2 |\mathcal{M}(\phi)|}{|\phi|}} \quad (5)$$

we denote the *entropy-based conciseness measure*, also called γ-*index*.

Note that by construction and Eq. 1 we also have

$$\gamma_h(\phi) = c_\phi \cdot \alpha_h(\phi)$$

and, thus,

$$0 \leq \gamma_h(\phi) \leq c_\phi = \frac{|\Sigma_\phi|}{|\phi|} \ .$$

By Propositions 21 and 23 it is immediate to show that

Proposition 43. *The function α_h (resp. γ_h) is an accuracy (resp. a conciseness) measure.*

Eventually, concerning Pareto optimality, Corollary 27 applies.

Example 44. *For*

$$\phi := p \to q$$

we have that

$$\mathcal{M}(\phi) \quad = \quad \{pq, \bar{p}q, \bar{p}\bar{q}\}$$

Therefore $|\mathcal{M}(\phi)| = 3$ and, thus,

$$\begin{aligned} \kappa_h(\phi) &= 2 - \log_2 3 \approx 0.42 \\ \alpha_h(\phi) &\approx 0.21 \\ \gamma_h(\phi) &\approx 0.14 \ . \end{aligned}$$

Example 45 (Example 24 cont.)**.** *Consider $\phi := p \land (p \to q)$. Then its κ_h-index is 2, its alphabet's size is 2, and its length is 5 and, thus, its α_h-index is 1, its γ_h-index is 0.4. Now, consider $\psi = p \land q$. As $\psi \equiv \phi$ we have $\kappa_h(\psi) = \kappa_h(\phi) = 2$, but $|\psi| = 3$ and, thus, ψ's γ_h-index is $\gamma_h(\psi) = 2/3 > 0.4 = \gamma_h(\phi)$, in agreement with Proposition 22. ψ's α_h-index is 1. In summary, ψ is equivalent to ϕ, so represents the same amount of knowledge, has the same accuracy, but is more concise than ϕ. More precisely, ψ is Pareto optimal (see Example 24).*

5 CONCLUSION & FUTURE RESEARCH DIRECTIONS

In this work we have introduced the notion of knowledge measure for propositional logic, *i.e.* κ-index, whose aim is to quantify the amount of knowledge a kb represents. To do so, we have defined four axioms $(T), (E), (L), (M)$ that a *knowledge measure* we believe should have[13]: (i) (T) establishes the conditions for tautology and contradiction; (ii) (E) tries to encode the fact that if a formula entails another one, then the former has more knowledge that the latter; (iii) (L) instead, states that symbols not occurring in a formula do not contribute to represent additional knowledge; and eventually (iv) axiom (M) provides us an upper bound and when the upper bound is attained.

We also introduce related kb notions such as accuracy, conciseness and Pareto optimality: the first one defines how precise a kb is in describing the actual world, the second one defines how succinct a kb is w.r.t. the knowledge it represents, while the last one establishes when we may not increase accuracy without decreasing conciseness (or vice-versa).

Eventually, we have provided a concrete example of such measures, based on the notion of entropy.

Of course, all measures can be implemented via propositional model counting. Even if counting the number of assignments satisfying a given propositional formula[14] is #P-complete, nowadays effective SAT solvers exists [6][15] .

Future research directions. Besides, of course, as in any other axiomatic approach, one may discuss about the appropriateness of the proposed axioms, we believe it is interesting to investigate on how one may extend the notion of κ-index to other logics: notably, the family of rule-like languages, modal logics, non-monotone logics, probabilistic/possibilistic logics, many/multiple-valued and fuzzy logics, all of them in the propositional as well as in the more challenging First-Order Logic setting. Another interesting problem is, given ϕ, to find (efficiently) a Pareto optimal solution w.r.t. ϕ. Last, but not least, we will investigate whether we may take advantage of what has been developed within the context of *philosophy of information* (see, e.g. [1, 2]).

REFERENCES

[1] *Philosophy of Information*, eds., Pieter Adriaans and Johan van Benthem, volume 8 of *Handbook of the Philosophy of Science*, Elsevier, 2008.
[2] Luciano Floridi, *The Philosophy of Information*, Oxford University Press, 2011.
[3] *Handbook of logic in artificial intelligence and logic programming (vol. 3): nonmonotonic reasoning and uncertain reasoning*, eds., Dov M. Gabbay, C. J. Hogger, and J. A. Robinson, Oxford University Press, Inc., New York, NY, USA, 1994.
[4] *Handbook of logic in artificial intelligence and logic programming (volumes 1 - 5)*, eds., Dov M. Gabbay, C. J. Hogger, and J. A. Robinson, Oxford University Press, Inc., New York, NY, USA, 1994/1998.
[5] Peter Gärdenfors, *Belief Revision*, Cambridge University Press, 2003.
[6] Carla P. Gomes, Ashish Sabharwal, and Bart Selman, 'Model counting', in *Handbook of Satisfiability*, eds., A. Biere, M. Heule, H. Van Maaren, and T. Walsh, volume 185 of *Frontiers in Artificial Intelligence and Applications*, chapter 20, IOS Press, (2009).
[7] S. Gottwald, *A Treatise on Many-Valued Logics*, A Research Studies Press Book, 2000.
[8] Petr Hájek, *Metamathematics of Fuzzy Logic*, Kluwer, 1998.

[13] We may coin it the $TELM$ system.
[14] Recall that the problem of counting the models of a formula is called the #SAT problem.
[15] http://beyondnp.org/pages/solvers/model-counters-exact/

[9] Darrel Hankerson, Peter D. Johnson, and Greg A. Harris, *Introduction to Information Theory and Data Compression*, CRC Press, Inc., Boca Raton, FL, USA, 1st edn., 1998.

[10] George J. Klir and Bo Yuan, *Fuzzy sets and fuzzy logic: theory and applications*, Prentice-Hall, Inc., Upper Saddle River, NJ, USA, 1995.

[11] Jack Minker, ‘On indefinite data bases and the closed world assumption’, in *Proc. of the 6th Conf. on Automated Deduction (CADE-82)*, ed., Springer-Verlag, volume 138 of *Lecture Notes in Computer Science*, (1982).