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New measures of inclusion between fuzzy sets in terms of the φ-index of inclusion

Nicolás Madrid and **Manuel Ojeda-Aciego**[1]

Abstract. The notion of *inclusion* is one of the most basic relations between sets, however, there is not a consensus about how to extend such a notion in fuzzy set theory. We introduce an alternative approach to previous methods in the literature in which we make use of the so-called φ-index of inclusion. This approach has a main difference with respect to previous ones: instead of a value in $[0, 1]$, the measure of inclusion is identified with a function. In this paper, using the φ-index of inclusion we define two measures of inclusion in the standard sense, i.e., taking a value in $[0, 1]$ and then, we show that both measures are in accordance with the standard axiomatic approaches about measures of inclusion in the literature.

1 INTRODUCTION

The information perceived and provided by humans is often imperfect; either vague, imprecise, uncertain, incomplete, etc. Accordingly, it is necessary the use of specific techniques capable to process imperfect data for the development of AI-tools oriented to the interaction with humans. In this respect, expert systems founded on fuzzy logic [41] have shown to be capable to deal suitably with such a kind of information. In general, expert systems are based on IF-THEN rules, which can be interpreted and applied from different perspectives. Here we can do a first distinction between two main families of approaches, those based on determining satisfiability degrees of the antecedent of rules (as the well-known Mamdani [30], Takagi-Sugeno [34] or Fuzzy Logic Programming [35] inference systems) and those based on determining a similarity or inclusion degree of the input with the antecedents of rules (as originally proposed by Zadeh [40] and axiomatized by Baldwin-Pilsworth [1] and Fukami [16]). This paper is of interest for the latter group of approaches that require the use of measures of inclusion in the inference systems [9, 19, 36].

Although the notion of *inclusion* is one of the most basic relations between sets, currently there is not a consensus about how to extend such a notion in fuzzy set theory. Possibly, the best known definition for inclusion is the original one provided by Zadeh in [41], which identifies inclusion between fuzzy sets with the point-wise ordering between membership functions. However, some approaches have criticized such a definition *"for being rigid and for the lack of softness according to the spirit of fuzzy logic"* (quoted from [10]). Basically, one can find three main kinds of approaches in the literature: those based on cardinality [11, 18, 22]; those based on logic implications [2, 5, 15]; and those based on axiomatic definitions [4, 12, 14, 21, 39]. Defining measures of inclusion is not only of theoretical interest since, for instance, in a framework of Social Science, fuzzy inclusion can be linked with mainstream statistical techniques [32], in a framework of data analysis, with classifiers [24] and the search of redundancy [25], and in a framework of image processing, with fuzzy mathematical morphology [13] and image quality measures [17].

We revisit here the φ-index[2], which was introduced in [29] originally to quantify the inclusion of a fuzzy set into another by following the guiding motto from [22]: *"A 'good' measure of inclusion should measure violations of Zadeh's inclusion"*; the main difference with respect to standard approaches to fuzzy inclusion is that, instead of representing the inclusion by a value in $[0, 1]$, the φ-index of inclusion is a mapping from $[0, 1]$ to $[0, 1]$. In [28], it was proved that φ-indexes of inclusion have properties which resemble those of some well-known measures of inclusion [27], and were used to define a new type of fuzzy similarity relations [28].

In this paper, we define two natural measures of inclusion in terms of the φ-index of inclusion, so that we can fairly compare its properties with those of other approaches to the inclusion between fuzzy sets. Specifically, we construct a measure of inclusion that satisfies the three axioms of Fan-Xie-Pie [14] and most of the axioms of Kitainik [21] and Sinha-Dougherty [31]; the other measure of inclusion satisfies the four axioms of Young [39] and, hence, also those of Fan-Xie-Pie.

The paper is structured as follows. Firstly, in Section 2 we recall the four most relevant axiomatic approaches about inclusion measures between fuzzy sets; namely, Kitainik, Sinha-Dougherty, Young and Fan-Xie-Pie approaches, then we finish the section by recalling the φ-index of inclusion and its main properties. In Section 3, we define two novel measures of inclusion, and show some of their properties and the relationships with the mentioned axiomatic approaches. Finally, in Section 4 we provide some conclusions and prospects for future works.

2 Preliminaries

A fuzzy set A is a pair $(\mathcal{U}, \mu_A)$ where $\mathcal{U}$ is a non-empty set (called the universe of A) and μ_A is a mapping from $\mathcal{U}$ to $[0, 1]$ (called membership function of A). In general, the universe is a fixed set for all the fuzzy sets considered and therefore, each fuzzy set is determined by its membership function. Hence, for the sake of clarity, we identify fuzzy sets with membership functions (i.e., $A(u) = \mu_A(u)$).

On the set of fuzzy sets defined on the universe $\mathcal{U}$, denoted $\mathcal{F}(\mathcal{U})$, we can extend the usual crisp operations of union, intersection and complement as follows. Given two fuzzy sets A and B, we define

- (union) $(A \cup B)(u) = \max\{A(u), B(u)\}$

[1] Universidad de Málaga. Dept. Matemática Aplicada, Spain, email: {nicolas.madrid,aciego}@uma.es

[2] The prefix φ- indicates that these indexes are *functional* parameters.

- (intersection) $(A \cap B)(u) = \min\{A(u), B(u)\}$
- (complement) $A^c(u) = 1 - A(u)$.

The previous extensions of union, intersection and complement are the most common in the literature but, certainly, there are other options. For example, as generalization of the previous extensions, many authors use t-norms to generalize intersection, t-conorms to generalize union and negation operators to generalize the complement.

Any transformation in the universe $T: \mathcal{U} \to \mathcal{U}$ can be extended to $\mathcal{F}(\mathcal{U})$ by defining for each $A \in \mathcal{F}(\mathcal{U})$ the fuzzy set $T(A)(u) = A(T(u))$.

In the rest of this section we recollect four established approaches for the axiomatic definition of measures of inclusion; namely, those of Sinha-Dougherty, of Kitainik, of Young approach and of Fan-Xie-Pei. It is worth mentioning that there are more measures of inclusion in the literature than those recalled here, like Kosko and DI-subsethood measures [23, 7], which are particular cases of Young measures; the inclusion grade [5] which is a particular case of the Sinha-Dougherty inclusion measure; or the Kaburlasos inclusion measure [20], which is a particular case of the Fan-Xie-Pei inclusion measure. Summarizing, the four axiomatic measures of inclusion recalled below are, under our point of view, the most important axiomatic definitions in the literature.

At the end of this section, we recall also the index of inclusion presented in [28] which, contrariwise to the previous approaches, intends to represent the inclusion between two fuzzy sets by means of functions instead of by values in $[0, 1]$.

2.1 Kitainik axioms

In 1987, Leonid Kitainik [21] proposed an axiomatic definition for measures of inclusion aimed at capturing those inclusion measures based on the minimum of implications; this approach was extensively applied during the eighties [2, 37].

Definition 1 *A mapping $\mathcal{I}: \mathcal{F}(\mathcal{U}) \times \mathcal{F}(\mathcal{U}) \to [0, 1]$ is called a* K-measure of inclusion *if it satisfies the following axioms for all fuzzy sets A, B and C:*

(K1) $\mathcal{I}(A, B) = \mathcal{I}(B^c, A^c)$.
(K2) $\mathcal{I}(A, B \cap C) = \min\{\mathcal{I}(A, B), \mathcal{I}(A, C)\}$.
(K3) *If $T: \mathcal{U} \to \mathcal{U}$ is a bijective transformation on the universe, then* $\mathcal{I}(A, B) = \mathcal{I}(T(A), T(B))$.
(K4) *If A and B are crisp then $\mathcal{I}(A, B) = 1$ if and only if $A \subseteq B$.*
(K5) *If A and B are crisp then $\mathcal{I}(A, B) = 0$ if and only if $A \nsubseteq B$.*

In [15], Fodor and Yager showed that, for every K-measure of inclusion $\mathcal{I}$, there exists a fuzzy implication $\to$ such that for all fuzzy sets A and B, the following equality holds

$$\mathcal{I}(A, B) = \bigwedge_{u \in \mathcal{U}} A(u) \to B(u) \,.$$

In other words, the axiomatic definition of Kitainik characterizes the measures of inclusion based on infimum of fuzzy implications.

2.2 Sinha-Dougherty axioms

One of the measures of inclusion most used in the literature was proposed by Divyendu Sinha and Edward R. Dougherty in 1993 [31]. Originally, they required nine axioms, but one could be inferred from the others (see [10, 5]) and then, we present here only the independent eight ones.

Definition 2 *A mapping $\mathcal{I}: \mathcal{F}(\mathcal{U}) \times \mathcal{F}(\mathcal{U}) \to [0, 1]$ is called an* SD-measure of inclusion *if it satisfies the following axioms for all fuzzy sets A, B and C:*

(SD1) $\mathcal{I}(A, B) = 1$ *if and only if $A(u) \leq B(u)$ for all $u \in \mathcal{U}$.*
(SD2) $\mathcal{I}(A, B) = 0$ *if and only if there exists $u \in \mathcal{U}$ such that $A(u) = 1$ and $B(u) = 0$.*
(SD3) *If $B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then $\mathcal{I}(A, B) \leq \mathcal{I}(A, C)$.*
(SD4) *If $B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then $\mathcal{I}(C, A) \leq \mathcal{I}(B, A)$.*
(SD5) *If $T: \mathcal{U} \to \mathcal{U}$ is a bijective transformation on the universe, then $\mathcal{I}(A, B) = \mathcal{I}(T(A), T(B))$.*
(SD6) $\mathcal{I}(A, B) = \mathcal{I}(B^c, A^c)$.
(SD7) $\mathcal{I}(A \cup B, C) = \min\{\mathcal{I}(A, C), \mathcal{I}(B, C)\}$.
(SD8) $\mathcal{I}(A, B \cap C) = \min\{\mathcal{I}(A, B), \mathcal{I}(A, C)\}$.

Although the definition of Sinha and Dougherty was introduced independently from the Kitainik approach, both are somewhat similar in that they share certain features. In fact, it is straightforward to prove that every SD-measure of inclusion is also a K-measure of inclusion.

2.3 Young axioms

The axioms proposed by Virginia R. Young [39] in 1996 focus on mesures of inclusion capable to define entropy measures [22] in terms of the value of the inclusion degree of $A \cup A^c$ into $A \cap A^c$. This fact generates an evident difference of the Young approach with respecto to the Kitainik and Sinha-Dougherty approaches.

Definition 3 *A mapping $\mathcal{I}: \mathcal{F}(\mathcal{U}) \times \mathcal{F}(\mathcal{U}) \to [0, 1]$ is called a* Y-measure of inclusion *if it satisfies the following axioms for all fuzzy sets A, B and C:*

(Y1) $\mathcal{I}(A, B) = 1$ *if and only if $A(u) \leq B(u)$ for all $u \in \mathcal{U}$.*
(Y2) *If $A(u) \geq 0.5$ for all $u \in \mathcal{U}$, then $\mathcal{I}(A, A^c) = 0$ if and only if $A = \mathcal{U}$; i.e., $A(u) = 1$ for all $u \in \mathcal{U}$.*
(Y3) *If $A(u) \leq B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then $\mathcal{I}(C, A) \leq \mathcal{I}(B, A)$.*
(Y4) *If $B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then $\mathcal{I}(A, B) \leq \mathcal{I}(A, C)$ for all fuzzy set $A \in \mathcal{F}(\mathcal{U})$.*

In the original definition [39], axioms (Y3) and (Y4) are stated jointly as one axiom but here we have preferred to write them separately for a better comparison with the other axiomatic approaches. Young shows that her axiomatic definition covers those measures of inclusion defined as the mean of implication operators [38].

2.4 Fan-Xie-Pei axioms

The definition of Young was analyzed and slightly modified by Jiulun Fan, Weixin Xie and Jihong Pie [14]. As a result, they propose three other different definitions of measure of inclusion called, respectively, strong measure of inclusion, measure of inclusion and weak measure of inclusion.

Definition 4 *A mapping $\mathcal{I}: \mathcal{F}(\mathcal{U}) \times \mathcal{F}(\mathcal{U}) \to [0, 1]$ is said to be a* strong FXP-inclusion measure *if it satisfies the following axioms for all fuzzy sets A, B and C:*

(sFXP1) $\mathcal{I}(A, B) = 1$ *if and only if $A(u) \leq B(u)$ for all $u \in \mathcal{U}$.*
(sFXP2) *If $A \neq \emptyset$ and $A \cap B = \emptyset$ then, $\mathcal{I}(A, B) = 0$.*
(sFXP3) *If $A(u) \leq B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then $\mathcal{I}(C, A) \leq \mathcal{I}(B, A)$ and $\mathcal{I}(A, B) \leq \mathcal{I}(A, C)$.*

Due to the restrictive condition of axiom (sFXP2), Fan, Xie and Pie also proposed the following definitions.

Definition 5 *A mapping $\mathcal{I}\colon \mathcal{F}(\mathcal{U}) \times \mathcal{F}(\mathcal{U}) \to [0,1]$ is said to be a* FXP-inclusion measure *if it satisfies the following axioms for all fuzzy sets A, B and C:*

(FXP1) $\mathcal{I}(A,B) = 1$ *if and only if* $A(u) \leq B(u)$ *for all* $u \in \mathcal{U}$.
(FXP2) $\mathcal{I}(\mathcal{U}, \emptyset) = 0$.
(FXP3) *If* $A(u) \leq B(u) \leq C(u)$ *for all* $u \in \mathcal{U}$ *then* $\mathcal{I}(C,A) \leq \mathcal{I}(B,A)$ *and* $\mathcal{I}(A,B) \leq \mathcal{I}(A,C)$.

Definition 6 *A mapping $\mathcal{I}\colon \mathcal{F}(\mathcal{U}) \times \mathcal{F}(\mathcal{U}) \to [0,1]$ is said to be* weak FXP-inclusion measure *if it satisfies the following axioms for all fuzzy sets A, B and C:*

(wFXP1) $\mathcal{I}(\emptyset,\emptyset) = \mathcal{I}(\emptyset,\mathcal{U}) = \mathcal{I}(\mathcal{U},\mathcal{U}) = 1$*; where* $\mathcal{U}(u) = 1$ *for all* $u \in \mathcal{U}$.
(wFXP2) $\mathcal{I}(\mathcal{U}, \emptyset) = 0$
(wFXP3) *If* $A(u) \leq B(u) \leq C(u)$ *for all* $u \in \mathcal{U}$ *then* $\mathcal{I}(C,A) \leq \mathcal{I}(B,A)$ *and* $\mathcal{I}(A,B) \leq \mathcal{I}(A,C)$.

In the original paper of Fan, Xie and Pie [14] the reader can find relationships between these measures and fuzzy implications.

2.5 The φ-index of inclusion

The idea behind the φ-index of inclusion is to quantify the inclusion of one fuzzy set into another by a mapping from $[0,1]$ to $[0,1]$, instead of a value in $[0,1]$ as the previous axiomatic measures of inclusion. The set of possible assignable mappings is called set of φ-indexes of inclusion, is denoted by Ω, and consists of every monotonically increasing mapping $f\colon [0,1] \to [0,1]$ such that $f(x) \leq x$ for all $x \in [0,1]$. In order to define the φ-index of inclusion, we need to introduce firstly the notion of f-inclusion [29].

Definition 7 *Let A and B be two fuzzy sets and consider $f \in \Omega$. We say that A* is f-included in *B (denoted by $A \subseteq_f B$) if and only if the inequality $f(A(u)) \leq B(u)$ holds for all $u \in \mathcal{U}$.*

Note that, fixed $f \in \Omega$, the relation of f-inclusion is a crisp relation and, in general, is not even an ordering relation (transitivity fails). Therefore, at first view the f-inclusion (with a fixed $f \in \Omega$) seems to be unsuitable to represent the inclusion between to fuzzy sets, since it is lacking of softness, like Zadeh's inclusion. However, we do not define the φ-index of inclusion by fixing an f-inclusion, but we consider all of them as different degrees of inclusion. Specifically, since each f-inclusion is determined by a mapping f in Ω, we consider mappings in Ω as indexes of inclusions. Such a consideration is described and motivated in detail in [29] and can be summarized in the following items:

- Ω has the structure of complete lattice with the natural ordering between functions; i.e., given $f, g \in \Omega$, we say that $f \leq g$ if $f(x) \leq g(x)$ for all $x \in [0,1]$. In particular, the mappings id (defined by $id(x) = x$ for all $x \in [0,1]$) and $\perp$ (defined by $\perp(x) = 0$ for all $x \in [0,1]$) are the top and bottom elements in Ω, respectively.
- Each $f \in \Omega$ determines a restriction, via the corresponding f-inclusion, that can be understood as "*how much do we have to reduce the truth values of a fuzzy set in order to be included into another in Zadeh's sense"*. Thus, each $f \in \Omega$ can be seen as how much Zadeh's inclusion is violated.
- Finally, the greater the mapping $f \in \Omega$ the stronger the restriction imposed by the f-inclusion. In particular, the id-inclusion is the most restrictive case (and is equivalent to Zadeh's inclusion) and the $\perp$-inclusion does not establish any restriction at all (below we show that $\perp$-inclusion is identified with no inclusion).

The φ-index of inclusion is based on the idea "the more f-inclusions holding between two sets, the greater is the inclusion". Fortunately, we do not need to check all the f-inclusions between two sets thanks to the following lemmas.

Lemma 1 *Let A and B be two fuzzy sets and let $f, g \in \Omega$ such that $f \leq g$. Then, $A \subseteq_g B$ implies $A \subseteq_f B$.*

Lemma 2 *Let A and B be two fuzzy sets and consider a family $\{f_i\}_{i \in I} \subseteq \Omega$. If A is f_i-included in B for all $i \in I$, then A is $\bigvee_{i \in I} f_i$-included in B.*

As a direct consequence of the previous two lemmas, given two fuzzy sets, the subset $\Lambda(A,B) = \{f \in \Omega \mid A \subseteq_f B\}$ has a maximum element in Ω. This fact allows us to introduce the following definition .

Definition 8 (φ-index of inclusion) *Let A and B be two fuzzy sets, the* φ-index of inclusion *of A in B, denoted by $Inc(A,B)$, is defined as the maximum of $\Lambda(A,B)$.*

Note firstly that the φ-index of inclusion of A in B does not depend on any prior assumption or any kind of parameter [15]. Secondly, note that thanks to Lemma 1, the set $\Lambda(A,B)$ of mappings $f \in \Omega$ such A is f-included in B is characterized by $Inc(A,B)$, since:

$$\Lambda(A,B) = \{f \in \Omega \mid A \subseteq_f B\} = \{f \in \Omega \mid f \leq Inc(A,B)\}$$

In [28], the following analytical expression for $Inc(A,B)$ was found:

Theorem 1 *Let A and B be two fuzzy sets, then $Inc(A,B) = f_{A,B} \wedge id$, where*

$$f_{A,B}(x) = \bigwedge_{u \in \mathcal{U}} \{B(u) \mid x \leq A(u)\}.$$

We summarize below some properties that motivate the use of $Inc(A,B)$ as a suitable index of inclusion between two fuzzy sets.

Theorem 2 ([28]) *Let A, B and C be fuzzy sets,*

1. *(Full inclusion) $Inc(A,B) = id$ if and only if $A(u) \leq B(u)$ for all $u \in \mathcal{U}$.*
2. *(Null inclusion) $Inc(A,B) = \perp$ if and only if there is a set of elements in the universe $\{u_i\}_{i \in I} \subseteq \mathcal{U}$ such that $A(u_i) = 1$ for all $i \in I$ and $\bigwedge_{i \in I} B(u_i) = 0$.*
3. *(Pseudo transitivity) $Inc(B,C) \circ Inc(A,B) \leq Inc(A,C)$.*
4. *(Monotonicity) if $B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then, $Inc(C,A) \leq Inc(B,A)$.*
5. *(Monotonicity) if $B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then, $Inc(A,B) \leq Inc(A,C)$;*
6. *(Transformation Invariance) Let A and B be two L-fuzzy sets and let $T\colon \mathcal{U} \to \mathcal{U}$ be a transformation on $\mathcal{U}$, then $Inc(A,B) = Inc(T(A), T(B))$.*
7. *(Relationship with intersection) $Inc(A, B \cap C) = Inc(A,B) \wedge Inc(A,C)$.*

8. *(Relationship with union)* $Inc(A \cup B, C) = Inc(A, C) \wedge Inc(B, C)$.

The final property of φ-indexes of inclusion relates the adjoint property and the complement. In order to maintain this paper self-contained, let us recall the notion of *adjoint property*: given a complete lattice L, we say that a pair (f, g) of mappings $f, g \colon L \to L$ satisfy the adjoint property in L if

$$f(x) \le y \iff x \le g(y) \qquad \text{for all } x \in L$$

Theorem 3 *(Relationship with complement) Let (f, g) be an adjoint pair in the unit interval, and n and involute negation, then*

$$A \subseteq_f B \text{ if and only if } B^c \subseteq_{n \circ g \circ n} A^c.$$

3 NEW MEASURES OF INCLUSION BASED ON THE φ-INDEX OF INCLUSION

The reader can easily obverse the close relationship between the properties recalled in Theorem 2 and the axiomatic approaches given by Sinha-Dougherty, Kitainik, Young and Fan-Xie-Pie recalled in the previous section. First of all, note that the comparison may be unfair, since those mentioned approaches consider measures of inclusion that return real values but, instead, the φ-index of inclusion is a mapping. Anyway, by rewriting the respective axioms in functional terms, the reader can check that the only axioms that are not satisfied are (K1), (SD6), (Y2) and (sFXP2). In this section, we provide a new measure of inclusion in terms of φ-indexes, prove that it is a Fan-Xie-Pie measure of inclusion such that the axioms (K1) and (SD6) are satisfied with this new measure; finally, this measure is further tuned so that it is not only a Fan-Xie-Pie measure but also a Young measure.

3.1 A new Fan-Xie-Pie measure

We know by Theorem 2 that the greater the φ-index of inclusion, the greater the inclusion. On the other hand, the best way to measure a function is by integrals. These two facts lead us to the following definition of measure of inclusion.

Definition 9 *Let A and B be two fuzzy sets, the φ-*measure of inclusion *of A in B, denoted by $M_{Inc}(A, B)$, is defined as*

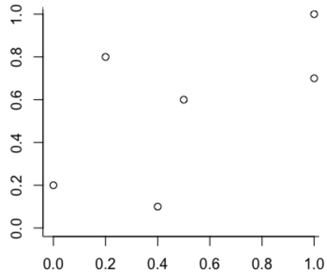
$$M_{Inc}(A, B) = 2 \int_0^1 Inc(A, B)(x)dx$$

The scalar 2 in the previous definition is included only for a normalization purpose; i.e., the maximum of this measure should be 1. Note also that the stronger the inclusion, the greater $Inc(A, B)$ and, then, the greater the value of $M_{Inc}(A, B)$ as well. The following example shows how the φ-measure of inclusion can be computed in terms of a graphical representation. Such a graphical representation also illustrates a meaning of M_{Inc}.

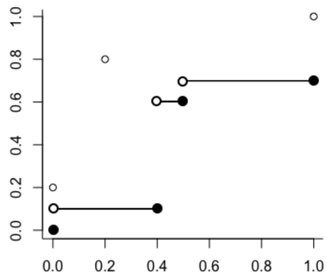
Example 1 *Let us consider the universe $\{u_0, u_1, u_2, u_3, u_4, u_5\}$ and the two fuzzy sets given by the table:*

$\mathcal{U}$	A	B
u_0	1	0.7
u_1	0.2	0.8
u_2	0	0.2
u_3	1	1
u_4	0.4	0.1
u_5	0.5	0.6

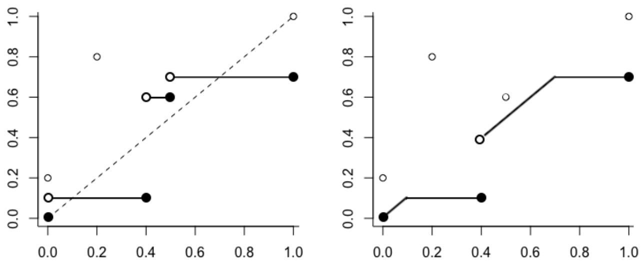
In order to compute the analytical expression of $Inc(A, B)$ we apply Theorem 1; hence, firstly, we have to compute the expression of the function $f_{A,B}$. To facilitate such a task, we represent in the unit square $[0, 1]^2$ the point $\{(A(u), B(u)) \mid u \in \mathcal{U}\}$; i.e., the truth degrees of elements in A and B.



Then, the graph of the function $f_{A,B}$ is easily determinable



and as result, the expression $Inc(A, B) = f_{A,B} \wedge id$ is directly achievable:

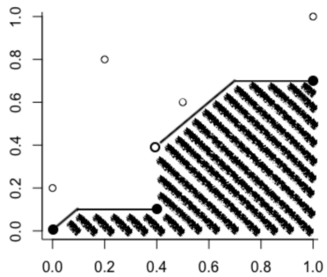


$$f_{A,B}(x) = \begin{cases} x & \text{if } x \le 0.1 \\ 0.1 & \text{if } 0.1 < x \le 0.4 \\ x & \text{if } 0.4 < x \le 0.7 \\ 0.7 & \text{if } 0.7 < x \le 1 \end{cases}$$

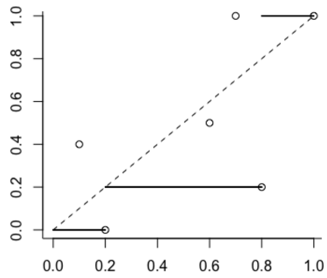
Finally, the measure of inclusion of A into B is given by

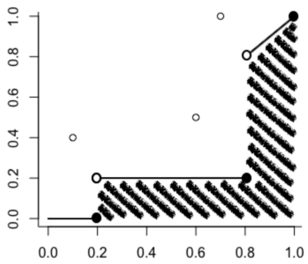
$$M_{Inc}(A, B) = 2 \int_0^1 Inc(A, B)(x)dx = 0.82$$

which represents the double of the area below $Inc(A, B)(x)$:



Note that the measure of inclusion is not symmetric. Following a similar procedure than above, we determine firstly the φ-index of inclusion $Inc(B, A)$:





and then, the measure of inclusion is

$$M_{Inc}(B, A) = 2\int_0^1 Inc(A, B)(x)dx = 0.68.$$

Therefore, we can conclude that A is more included in B than B in A under the measure M_{Inc}.

As expected, the measure of inclusion M_{Inc} is a FXP-measure of inclusion.

Theorem 4 *M_{Inc} is a FXP-measure of inclusion, that is, for all fuzzy sets A, B and C:*

- $M_{Inc}(A, B) = 1$ *if and only if* $A(u) \leq B(u)$ *for all* $u \in \mathcal{U}$.
- $M_{Inc}(\mathcal{U}, \emptyset) = 0$.
- *If* $A(u) \leq B(u) \leq C(u)$ *for all* $u \in \mathcal{U}$ *then* $M_{Inc}(C, A) \leq M_{Inc}(B, A)$ *and* $M_{Inc}(A, B) \leq M_{Inc}(A, C)$.

As a consequence of the previous theorem, M_{Inc} is also a weak FXP-measure of inclusion. However, M_{Inc} is not a strong FXP-measure of inclusion, since axiom (sFXP2) of Definition 4 does not hold; as the following example shows.

Example 2 *Consider the universe $\mathcal{U} = \{u_0, u_1\}$ and the fuzzy sets A and B defined by $A(u_0) = 0$, $A(u_1) = 0.5$, $B(u_0) = 0.5$ and $B(u_1) = 0$. In order to compute the analytical expression of $Inc(A, B)$ we apply Theorem 1 as in Example 1. We firstly compute the expression of the function $f_{A,B}$:*

$$f_{A,B}(x) = \bigwedge_{u\in\mathcal{U}} \{B(u) \mid x \leq A(u)\} = \begin{cases} 0 & \text{if } x \leq 0.5 \\ 1 & \text{otherwise} \end{cases}$$

and then,

$$Inc(A, B)(x) = f_{A,B}(x) \wedge id = \begin{cases} 0 & \text{if } x \leq 0.5 \\ x & \text{otherwise} \end{cases}$$

Note that both A and B are different from $\emptyset$ but $A \cap B = \emptyset$, however,

$$M_{Inc}(A, B) = 2\int_0^1 Inc(A, B)(x)dx = 2\int_{0.5}^1 xdx = 0.75$$

Therefore $M_{Inc}(A, B) \neq 0$ and M_{Inc} does not satisfy axiom (sFXP2).

The axiom (sFXP2) in Definition 4 intends to represent the relationship between the empty set and the measure of inclusion. However, under our point of view, the relationship modelled is quite drastic and goes against the inherent gradualness of fuzzy sets. For instance, we can consider $B = \emptyset$ and a fuzzy set A as close to the empty set as desired. In all those cases, the axiom (sFXP2) implies that the inclusion of A in B is 0, but our intuition says something different, since the closer is A to B, the greater should be the inclusion of A in B.

Note that in the crisp setting, "being different from the empty set" is equivalent to assert the existence of an element in the set. However, both ideas are generalized differently in the fuzzy setting, whereas the first statement is generalized as $A \neq \emptyset$ (as Fan, Xie and Pie do in the axiom (sFXP2)), the latter statement is generalized by the notion of normality (i.e., a fuzzy set A is normal if there is an element $u \in \mathcal{U}$ such that $A(u) = 1$). Changing "$A \neq \emptyset$" by "A is normal" in axiom (sFXP2) we obtain the corresponding property.

Proposition 1 *Let A and B be two fuzzy sets. If A is normal and $A \cap B = \emptyset$, then $M_{Inc}(A, B) = 0$.*

The previous result can be generalized in a more complex but convenient way: a smooth transition of the hypothesis from $\emptyset$ to normality implies a smooth transition of an upper bound for the measure M_{Inc}, which is clearly in accordance to the spirit of fuzzy sets.

Proposition 2 *Let A and B be two fuzzy sets and let $\alpha, \beta \in [0, 1]$ such that $\beta \leq \alpha$. If there exists $u_0 \in \mathcal{U}$ such that $A(u_0) \geq \alpha$ and $A \cap B(u) \leq \beta$ for all $u \in \mathcal{U}$, then*

$$M_{Inc}(A, B) \leq 1 - (\alpha - \beta)^2.$$

Note that the hypothesis of the previous result can be interpreted as follows: on the one hand, the existence of $u_0 \in \mathcal{U}$ such that $A(u_0) \geq \alpha$ is like a degree of normality of A and on the other hand, $A \cap B(u) < \beta$ for all $u \in \mathcal{U}$ is like a degree of emptiness of $A \cap B$. Hence, the previous result can be roughly paraphrased in natural language as "*the more normal the fuzzy set A and the more empty the intersection $A \cap B$, the lesser the inclusion of A in B*". That is, the measure of inclusion M_{Inc} satisfies a fuzzy version of the axiom (sFXP2).

In the rest of the paper, we focus on the properties of M_{Inc} oriented to the axiomatic definitions of Sinha-Dougherty and Kitainik.

To begin with, we have that M_{Inc} satisfies axiom (SD1).

Proposition 3 *Let A and B be two fuzzy sets. $M_{Inc}(A, B) = 1$ if and only if $A(u) \leq B(u)$ for all $u \in \mathcal{U}$.*

As a consequence of the previous proposition, the axiom (K4) also holds. Concerning the axioms (SD2) and (K5) we have the following result. Note that the statement resembles item 2 of Theorem 2 but there are differences between them, though.

Theorem 5 *Let A and B be two fuzzy sets. $M_{Inc}(A, B) = 0$ if and only if there is a set of elements in the universe $\{u_i\}_{i\in I} \subseteq \mathcal{U}$ such that $\bigvee_{i\in I} A(u_i) = 1$ and $\bigwedge_{i\in I} B(u_i) = 0$.*

Note that, as a consequence of the previous result, the axiom (K5) is satisfied by the measure of inclusion M_{Inc}.

Corollary 1 *Let A and B be two crisp sets. $M_{Inc}(A, B) = 0$ if and only if there exists $u \in \mathcal{U}$ such tat $A(u) = 1$ and $B(u) = 0$.*

Another consequence of Theorem 5 is that the axiom (SD2) is satisfied by the measure of inclusion M_{Inc} when the underlying universe considered is finite.

Corollary 2 *Let A and B be two fuzzy sets defined on a finite universe $\mathcal{U}$. $M_{Inc}(A, B) = 0$ if and only if there exists $u \in \mathcal{U}$ such tat $A(u) = 1$ and $B(u) = 0$.*

The following result concerns the monotonicity of M_{Inc}.

Proposition 4 *Let A, B and C be three fuzzy sets such that $B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then:*

- $M_{Inc}(A, B) \leq M_{Inc}(A, C)$;
- $M_{Inc}(C, A) \leq M_{Inc}(B, A)$.

The transformation invariance of M_{Inc} is also a consequence of Theorem 2.

Proposition 5 *Let A and B be two fuzzy sets and $T : \mathcal{U} \to \mathcal{U}$ a bijective transformation on the universe $\mathcal{U}$. Then, $M_{Inc}(A, B) = M_{Inc}(T(A), T(B))$.*

Note that the only axiom of Sinha-Dougherty and Kitainink that is not satisfied[3] by the φ-index of inclusion in Theorem 2 is exactly the one concerned to complement of fuzzy sets. The following result shows that that axiom is satisfied by the measure of inclusion M_{Inc}.

Theorem 6 *Let A and B be two fuzzy sets, then $M_{Inc}(A, B) = M_{Inc}(B^c, A^c)$.*

Despite of all the properties stated above relating M_{Inc} to the axiomatic definitions of Sinha-Dougherty and Kitainik, it is convenient to remark that M_{Inc} is neither a SD-measure of inclusion nor a K-measure of inclusion. The reason in both cases is the same, M_{Inc} does not satisfy the axioms (SD7)=(K2) and (SD8). The following example shows a counterexample for both axioms.

Example 3 *Consider the universe $\mathcal{U} = \{u_0, u_1\}$ and the fuzzy sets A, B and C given by $A(u_0) = 0.5$, $A(u_1) = 1$, $B(u_0) = 1$, $B(u_1) = 0.5$, $C(u_0) = 0$ and $C(u_1) = 1$. It is easy to check that the φ-indexes of inclusion $Inc(A, B)$ and $Inc(A, C)$ are respectively:*

$$Inc(A, B) = \begin{cases} x & \text{if } x \leq 0.5 \\ 0.5 & \text{otherwise} \end{cases}$$

$$Inc(A, C) = \begin{cases} 0 & \text{if } x \leq 0.5 \\ x & \text{otherwise} \end{cases}$$

Moreover, by item 7 of Theorem 2 we have as a consequence:

$$Inc(A, B \cap C) = Inc(A, B) \wedge Inc(A, C) = \begin{cases} 0 & \text{if } x \leq 0.5 \\ 0.5 & \text{otherwise} \end{cases}$$

Then, we have the following corresponding measures of inclusion: $M_{Inc}(A, B) = M_{Inc}(A, C) = 0.75$ and $M_{Inc}(A, B \cap C) = 0.5$. Hence, the axiom (SD8) given by the equality $\mathcal{I}(A, B \cap C) = \min\{\mathcal{I}(A, B), \mathcal{I}(A, C)\}$ does not hold in this case.

On the other hand, let us consider in addition the fuzzy sets D and E given by: $D(u_0) = 0$, $D(u_1) = 0.5$, $E(u_0) = 0.5$ and $E(u_1) = 0$. Then, we have the following φ-indexes of inclusion:

$$Inc(C, D) = \begin{cases} x & \text{if } x \leq 0.5 \\ 0.5 & \text{otherwise} \end{cases}$$

$$Inc(E, D) = \begin{cases} 0 & \text{if } x \leq 0.5 \\ x & \text{otherwise} \end{cases}$$

and by item 8 of Theorem 2 we have:

$$Inc(C \cup E, D) = Inc(C, D) \wedge Inc(E, D) = \begin{cases} 0 & \text{if } x \leq 0.5 \\ 0.5 & \text{otherwise} \end{cases}$$

Finally, the corresponding measures of inclusion are: $M_{Inc}(C, D) = M_{Inc}(E, D) = 0.75$ but $M_{Inc}(C \cup E, D) = 0.5$. Hence, the axioms (SD7) and (K2) given by the equality $\mathcal{I}(C \cup E, D) = \min\{\mathcal{I}(C, D), \mathcal{I}(E, D)\}$ does not hold in this case.

Note that despite of the previous counter-example, the φ-index of inclusion actually gathers the idea behind axioms (SD8), (SD7) and (K2), which is visible in Theorem 2. Actually, Theorem 6 can be interpreted as, although the φ-index of inclusion does not satisfy directly the axioms (SD6) and (K1), the inclusion determined by the respective φ-index of inclusions "*measure the same inclusion*". As a result, the φ-index of inclusion covers all the underlying ideas behind both, the axiomatic definitions of Sinha-Dougherty and Kitainik.

3.2 From M_{Inc} to a Young measure of inclusion

The main difference between the axiomatic definition of Young with respect to both Kitainik and Sinha-Dougherty axiomatic approaches resides in Axiom (Y2), which is related to null measure and is contradictory with the respective axioms related to null measure (SD2) and (K5); quoting Virginia R. Young [39]: *"This axiom follows the spirit of Willmott [38], in which he defines a subsethood measure as a mean value of an implication operator"*. Following this idea, we propose a new measure of inclusion between fuzzy sets defined on finite universes as the mean of the point-wise inclusion measure given by M_{Inc}.

In order to make use of a point-wise inclusion measure, we have to do the following formal considerations: Given two fuzzy sets A and B, we plan to measure the inclusion of A in B for each element in the universe individually. Formally, for each $u \in \mathcal{U}$, we define the singleton universe $\mathcal{U}_u = \{u\}$ and the fuzzy sets $(\mathcal{U}_u, A_u)$ and $(\mathcal{U}_u, B_u)$ defined by $A_u(u) = A(u)$ and $B_u(u) = B(u)$. The *single measure of inclusion for $u \in \mathcal{U}$* is defined as $M_{Inc}(A_u, B_u)$. Once the notion of single measure of inclusion has been introduced, we can define the following measure of inclusion.

Definition 10 *Let A and B be two fuzzy sets defined on a finite universe. We define the mean-measure of inclusion as the value:*

$$\widehat{M}_{Inc}(A, B) = \frac{\sum_{u \in \mathcal{U}} M_{Inc}(A_u, B_u)}{Card(\mathcal{U})}$$

Theorem 7 *The mean-measure of inclusion $\widehat{M}_{Inc}$ is a Young measure of inclusion. That is:*

(Y1) *$\widehat{M}_{Inc}(A, B) = 1$ if and only if $A(u) \leq B(u)$ for all $u \in \mathcal{U}$*
(Y2) *if $A(u) \geq 0.5$ for all $u \in \mathcal{U}$, then $\widehat{M}_{Inc}(A, A^c) = 0$ if and only if $A(u) = 1$ for all $u \in \mathcal{U}$.*
(Y3) *If $A(u) \leq B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then $\widehat{M}_{Inc}(C, A) \leq \widehat{M}_{Inc}(B, A)$.*
(Y4) *If $B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then $\widehat{M}_{Inc}(A, B) \leq \widehat{M}_{Inc}(A, C)$ for all fuzzy set $A \in \mathcal{F}(\mathcal{U})$.*

Note that, as consequence of the previous theorem, $\widehat{M}_{Inc}$ is also a Fan-Xie-Pie measure of inclusion.

Corollary 3 *$\widehat{M}_{Inc}$ is a FXP-measure of inclusion.*

Besides $\widehat{M}_{Inc}$ is a Young measure of inclusion, it satisfies the contraposition law; that is, it satisfies the axiom (K1).

[3] Under the natural interpretation of axioms in the set $[0, 1]^{[0,1]}$.

Proposition 6 *Let A and B be two fuzzy sets, then $\widehat{M}_{Inc}(A,B) = \widehat{M}_{Inc}(B^c, A^c)$.*

Let us end the section with an example to illustrate the differences between the measures M_{Inc} and $\widehat{M}_{Inc}$.

Example 4 *Let us consider over the universe $\mathcal{U} = \{u_1, u_2, u_3\}$ the fuzzy sets A, B and C given by $A(u_1) = 1$, $A(u_2) = 0.5$, $A(u_3) = 1$, $B(u_1) = 0.1$, $B(u_2) = 0.2$, $B(u_3) = 0.4$, $C(u_1) = 0.1$, $C(u_2) = 0.6$ and $C(u_3) = 1$. Then, it is not difficult to check that $Inc(A,B)$ coincides with $Inc(A,C)$ and:*

$$Inc(A,B) = Inc(A,C) = \begin{cases} x & \text{if } 0 \le x \le 0.1 \\ 0.1 & \text{otherwise.} \end{cases}$$

Therefore, we have the measure of inclusion $M_{Inc}(A,B) = M_{Inc}(A,C) = 0.19$.

Let us calculate now the respective single measures of inclusion for u_1, u_2 and u_3. We have:

$$Inc(A_{u_1}, B_{u_1}) = \begin{cases} x & \text{if } 0 \le x \le 0.1 \\ 0.1 & \text{otherwise.} \end{cases}$$

$$Inc(A_{u_2}, B_{u_2}) = \begin{cases} x & \text{if } 0 \le x \le 0.2 \\ 0.2 & \text{if } 0.2 < x \le 0.5 \\ x & \text{otherwise.} \end{cases}$$

$$Inc(A_{u_3}, B_{u_3}) = \begin{cases} x & \text{if } 0 \le x \le 0.4 \\ 0.4 & \text{otherwise.} \end{cases}$$

$$Inc(A_{u_1}, C_{u_1}) = \begin{cases} x & \text{if } 0 \le x \le 0.1 \\ 0.1 & \text{otherwise.} \end{cases}$$

and $Inc(A_{u_2}, C_{u_2}) = Inc(A_{u_3}, C_{u_3}) = id$. As a result, we have the following measures:

$$\begin{array}{ll} M_{Inc}(A_{u_1}, B_{u_1}) = 0.19 & M_{Inc}(A_{u_2}, B_{u_2}) = 0.91 \\ M_{Inc}(A_{u_3}, B_{u_3}) = 0.64 & M_{Inc}(A_{u_1}, C_{u_1}) = 0.19 \\ M_{Inc}(A_{u_2}, C_{u_2}) = 1 & M_{Inc}(A_{u_3}, C_{u_3}) = 1 \end{array}$$

Then, the measures of inclusion are $\widehat{M}_{Inc}(A,B) = 0.58$ and $\widehat{M}_{Inc}(A,C) = 0.73$.

Note that although the measures of inclusion coincide with respect to M_{Inc}, they do not with respect to the measure $\widehat{M}_{Inc}$. The reason is that the measure of inclusion related to u_1 collapses the others in M_{Inc}; so the other truth values have not effect. However, in the measure of inclusion $\widehat{M}_{Inc}$ all the truth values are considered, and then, the single inclusions related to u_2 and u_3 are also taken into account.

4 Conclusions

The notion of φ-index, originally used to quantify the inclusion of a fuzzy set into another, is revisited. Two natural measures of inclusion in terms of the φ-index of inclusion are defined in order to compare its properties with those of other well-known approaches to the inclusion between fuzzy sets.

The two measures of inclusion introduced satisfy all the axioms of Fan-Xie-Pie and also the contrapositive law, contrariwise to the φ-index of inclusion in which this law does not hold. The contrapositive law make these new measures specially suitable for further advancing the theory of weak-contradictions [6] and the study of inconsistency in fuzzy answer-set semantics [26]. The difference between the two measures is their underlying philosophy. Whereas one is a Young measure of inclusion, the other is oriented to the Kitainik and Sinha-Dougherty (KSD) measures of inclusion. This fact opens two different future research lines. On the one hand, Young measures are closely related to entropy measures and have been applied to Decision making [8], whereas Kitainik and Sinha-Dougherty are related to fuzzy implications and then, to Knowledge-Based Systems and Inferences [5].

Furthermore, using the φ-index of inclusion Inc as a seed for the definition of different measures of inclusion, the consideration of φ-index of inclusion for other purposes will be considered. Note that the measures presented in this work are founded on the fact that the φ-index of inclusion represents a certain kind of inclusion between fuzzy sets. Then, the functional feature of the φ-index of inclusion is of great interest for frameworks where functional operators, similarities and inclusion are combined. One of those frameworks is might Fuzzy Logic Programming [35], which can be applied, for instance, to define ontologies in the framework of Fuzzy Description Logic [33, 3]. The inference system in fuzzy logic programming is based on rules IF-THEN associated to monotonic functions. We believe that the monotonic functions used in fuzzy logic programming can be related to the φ-index of inclusion Inc and, as a result, it would be possible to define inference procedures based on the φ-index of inclusion.

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