Lifting Majority to Unanimity in Opinion Diffusion

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Abstract. In this paper, we study an information exchange process in which a network of individuals exchanges a binary opinion. In the process, the individuals change their opinions only if a majority of their neighbours have the opposite opinion and they do it synchronously. Motivated by applications in multiagent systems, distributed computing, and social science, our goal is to derive graph-theoretic features of the network that guarantee whenever a majority of individuals initially have the same opinion, they will eventually spread the opinion to all individuals. We tackle the problem by first introducing a graph-theoretic notion called controlling set which is capable of characterising the information exchange process and, by exploiting the notion, we obtain a series of lower and upper bounds on the in-degree of vertices as well as lower bound on the size of certain neighbourhoods for guaranteeing the majority to unanimity behaviour.

1 Introduction

In this paper, we study an information exchange process often referred to as *opinion diffusion* [1, 2, 4, 5, 6, 7, 8, 9, 11, 12, 14] in *mutltiagent systems*. The process also goes by the name of *majority dynamics* in discrete mathematics [3, 18, 20, 21] and *general threshold model* in social influences [10, 13, 15]. With minor variations, the primitive form of the process is as follows. For a finite network of individuals, at time 0, each has an initial opinion of either 1 or 0, then at time $t \geqslant 1$, an individual i changes its opinion only if a majority of its neighbours have the opposite opinion. Depending on the problem setting, the individuals update their opinions either synchronously or asynchronously and the network is formalized as an undirected graph, a directed graph or other specialized graphs such as expander graph. Since there is a finite number of individuals, the process will either stabilize where no individual makes any change to its opinion or run into a cycle where the individuals periodically change their opinions.

The central problem of this paper is to identify network features that guarantee an opinion diffusion always stabilizes with all individuals having 1 (0) as their opinions, whenever a majority of the individuals initially have the opinion of 1 (resp. 0). That is, to ensure the initial majority opinion eventually become the unanimous opinion. For convenience, we refer to the problem as the *majority to unanimity problem*. To be clear, by 'majority,' we mean a sufficiently large proportion of the individuals where the proportion is specified by the threshold variable T such that $1/2 \leqslant T \leqslant 1$. Also, we formalize the network as a directed graph and adopt the synchronous mode of opinion update.

The majority to unanimity problem arises naturally in many application scenarios of multiagent systems, distributed computing, and social sciences. Whenever majority voting is adopted by a society to reach a consensus that represents the genuine majority opinion of the society, such as in a referendum, we need to guarantee the triumph of the initial majority opinion, as in the majority to unanimity problem. In the realm of discrete mathematics, a good deal of effort has been made in solving the majority to unanimity problem and its variants [3, 18, 20, 21]. It is pointed out in [21] that several aspects of distributed computing can be formulated as a majority to unanimity problem. For instance, to alleviate the damage caused by faulty processors, replicated copies of data are maintained. A simple majority voting process is performed among the participating processors whenever a fault occurs, and the goal is to adopt the initial value stored at the majority of the processors. The process almost always restores the correct value of the data, as it is believed that only a minority of processors can be faulty at any time. For this recovery process to work, the structure of the processor network has to ensure that a majority number of processors always take control of all processors in the voting process. Finally, the line of works in social sciences that aims to understand how opinions are formed and expressed in a social context also benefits from studying the majority to unanimity problem.

There have been many efforts in the multiagent system community to investigate behaviours of opinion diffusion related to the majority to unanimity problem. For instance, [14] articulates properties of the aggregation rules and features of the network that lead to the stabilization and convergence of opinion diffusion. Additionally, [2] aims to identify conditions for a minority/majority of individuals to spread their opinions to the whole population in terms of the configurations of the initial opinions. In contrast to our setting, they assume the individuals update their opinions asynchronously. The discrete mathematics community also investigated issues related to the majority to unanimity problem. Most noticeably, they study the extremal combinatorial aspects of opinion diffusion such as to establish lower bounds for the cardinality of an individual set capable of controlling the whole population in an opinion diffusion [3, 20, 21]. Moreover [18] tackles the majority to unanimity problem through the exploration of network structure. They provide several sufficient or necessary conditions on networks formalized as undirected graphs. Many results in this paper generalize or strengthen those in [18].

In addressing the majority to unanimity problem, we first introduce a graph-theoretic notion called *controlling set* which makes the collective influential power of a set of individuals to another in an opinion diffusion explicit. Roughly speaking, if a set of individuals C has a unified opinion and is a controlling set of another set of indi-

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viduals S, then the individuals of C can persuade those of S to agree with the unified opinion within some predefined rounds of opinion exchange. As we will show, the controlling relationship specified by the notion of controlling set fully characterizes the behaviour of opinion diffusion. Because of this, to obtain guaranteeing conditions for the majority to unanimity problem, instead of dealing with the iterated and dynamic process of opinion diffusion, we only have to look into the directed graph that formalizes the underlying network. Following this strategy, we derive a series of lower and upper bounds on the indegree of vertices of the directed graph and lower bound on the size of some neighbourhoods of the directed graph that form sufficient and/or necessary conditions for the majority to unanimity problem.

The formal model of opinion diffusion is presented in the next section. In Section 3, we introduce the notion of controlling set and demonstrate its capability of fully characterizing opinion diffusion. Then Section 4 covers a range of sufficient and/or necessary conditions formalized in graph-theoretic terms for the majority to unanimity problem. Finally, we give the related work and conclude the paper.

2 Opinion Diffusion

Opinion diffusion is a discrete sequence of synchronous opinion aggregation and update among a network of individuals. At any time $t \geqslant 1$, each individual aggregates the opinions of its influencers at time $t-1$ and update its opinion accordingly. We denote the opinion of individual i at t as O_i^t and the *opinion profile* at t as the n-dimensional vector $\mathbf{O}^t = \langle O_1^t, \ldots, O_n^t \rangle$ where n is the number of individuals in the network. O_i^0 and $\mathbf{O}^0$ denote respectively the *initial opinion* of i and the *initial opinion profile*.

Formally, an opinion diffusion $\mathcal{OD} = (\mathcal{G}, T, \mathbf{O}^0)$ consists of a finite directed graph $\mathcal{G}$, a threshold T, and an initial profile $\mathbf{O}^0$. The directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ represents the influence among a set of individuals where $\mathcal{N} = \{1, \ldots, n\}$ is a set of n vertices representing the individuals and $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ is a set of edges between pairs of vertices representing the existence of influence from one individual to another where $(i, j) \in \mathcal{E}$ indicates i is an *influencer* of j. The threshold T indicates the threshold proportion such that $1/2 \leqslant T < 1$. An individual alters its opinion at time t only if the proportion of its influencers having the opposite opinion at time $t-1$ is greater than T.

We denote the set of individuals that influence individual i as $I^-(i)$ and those influenced by i as $I^+(i)$, that is

$$I^-(i) = \{j \in \mathcal{N} \mid (j, i) \in \mathcal{E}\}$$

and

$$I^+(i) = \{j \in \mathcal{N} \mid (i, j) \in \mathcal{E}\}.$$

By abuse of notation, we write $\bigcup_{i \in S} I^+(i)$ as $I^+(S)$ and similarly for $I^-(S)$. Moreover, we denote the indegree of a vertex (i.e., an individual) i as $d(i)$, hence $d(i) = |I^-(i)|$. Note that, for any set S, $|S|$ is its cardinality. We may attach subscripts to $I^-(i)$ and $I^+(i)$ to denote their subsets and similarly to $d(i)$. For example, given a set of individuals C, $d_C(i) = |I_C^-(i)| = |I^-(i) \cap C|$. Finally, given an opinion profile $\mathbf{O}^t$, we denote the set of individuals whose opinions are 1 and respectively 0 in $\mathbf{O}^t$ as $\mathbf{1}_{\mathbf{O}^t}$ and $\mathbf{0}_{\mathbf{O}^t}$. Since $\mathbf{1}_{\mathbf{O}^t} = \mathcal{N} \setminus \mathbf{0}_{\mathbf{O}^t}$, $\mathbf{1}_{\mathbf{O}^t}$ or $\mathbf{0}_{\mathbf{O}^t}$ alone uniquely identifies the opinion profile $\mathbf{O}^t$. We say an opinion profile $\mathbf{O}^t$ has a *majority opinion* if either $|\mathbf{1}_{\mathbf{O}^t}|/n > T$ or $|\mathbf{0}_{\mathbf{O}^t}|/n > T$, otherwise it has no majority opinion.

The change of an individual's opinion is governed by the following *update rule*:

$$O_i^t = \begin{cases} 1 & \text{if } d_{\mathbf{1}_{\mathbf{O}^{t-1}}}(i)/d(i) > T \\ 0 & \text{if } d_{\mathbf{0}_{\mathbf{O}^{t-1}}}(i)/d(i) > T \\ O_i^{t-1} & \text{otherwise} \end{cases}$$

for all $i \in \mathcal{N}$ and $t \geqslant 1$. That is, an individual changes its opinion only if the proportion of its influencers having the opposite opinion exceeds T.

The two most common and indeed fundamental properties of opinion diffusion are *stabilization* and *convergence*. We say an opinion diffusion $\mathcal{OD} = (\mathcal{G}, T, \mathbf{O}^0)$ *stabilizes* at t if $\mathbf{O}^t = \mathbf{O}^{t+1}$ and we call $\mathbf{O}^t$ the stabilized opinion profile. Moreover, if the opinion diffusion stabilizes at t and the stabilized opinion profile is such that $O_1^t = O_2^t = \cdots = O_n^t$, we say the opinion diffusion *converges* at t.

Figure 1 illustrates an opinion diffusion with $T = 1/2$ and an initial opinion profile $\langle 1, 1, 1, 0, 0, 0, 0 \rangle$. Note that, all nodes in the graph have a self-loop which is omitted for clarity; a cycle involving two nodes is simplified to a two-way arrow, and a vertex filled by grey indicates the corresponding individual's opinion is 1 and otherwise 0. Starting from time 1, each individual updates its opinion according to the update rule. For example, at time 1, since individual 3 has more than half of its influencers having the opinion of 1, (i.e., $d_{\mathbf{1}_{\mathbf{O}^0}}(3)/d(3) = d_{\{1,2,3\}}(3)/d(3) = 3/4 > 1/2$), the individual updates its opinion to 1 which happens to be its initial opinion. One can verify that the opinion diffusion stabilizes and converges at time 2. For this opinion diffusion, starting with the initial majority opinion of 0, it stabilizes with all individuals having a unanimous opinion of 1. Clearly, this is what we want to avoid for the majority to unanimity problem. For the ease of presentation, we formalize the property of opinion diffusion with respect to the majority to unanimity problem as *majority dominating*.

Definition 1. *An opinion diffusion* $\mathcal{OD} = (\mathcal{G}, T, \mathbf{O}^0)$ *is* majority dominating *iff there is a time t such that* $\mathbf{O}^t = \langle 1, \ldots, 1 \rangle$ ($\mathbf{O}^t = \langle 0, \ldots, 0 \rangle$) *whenever* $|\mathbf{1}_{\mathbf{O}^0}|/n > T$ *(resp.* $|\mathbf{0}_{\mathbf{O}^0}|/n > T$*).*

That is, an opinion diffusion is majority dominating iff whenever there is an initial majority opinion the opinion diffusion converges on it. Recall that $|\mathbf{1}_{\mathbf{O}^0}|$ is the number of individuals whose initial opinion is 1, hence $|\mathbf{1}_{\mathbf{O}^0}|/n > T$ means 1 is the initial majority opinion. The opinion diffusion in Figure 1 is not majority dominating, because, although it converges, the opinion that everyone has at convergence (i.e., 1) is different from the initial majority opinion (i.e., 0).

Before ending this section, we give the complexity result for determining whether an opinion diffusion is stabilizable, convergent, and majority dominating respectively.

Proposition 1. *It is PSPACE-complete to determine whether a given opinion diffusion is stabilizable, convergent, and majority dominating respectively.*

Proof. The membership is trivial for all the desired cases. Here we only consider the hardness. The PSPACE-hardness of stabilizability immediately follows from Theorem 1 in [6] which asserts that deciding whether a given opinion diffusion with threshold $1/2$ is stabilizable is PSPACE-complete. Below we show how to prove the PSPACE-hardness of majority dominating. Note that the case of convergence can be proved by almost the same argument.

The general idea is to reduce the problem of checking whether a given polynomial space-bounded Turing machine halts to our problem (i.e., checking whether a given opinion diffusion is majority

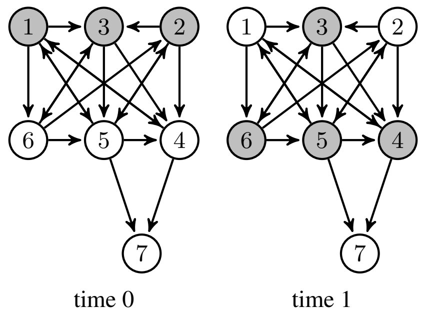

Figure 1. An opinion diffusion with $\mathbf{O}^0 = \langle 1, 1, 1, 0, 0, 0, 0 \rangle$ and $T = 1/2$.

dominating). With this reduction, as the former problem was proved to be PSPACE-hard (see Theorem 19.9 in [19] and the proof of Theorem 1 in [6]), we thus obtain the desired hardness.

Now it remains to implement the reduction. Given any polynomial space-bounded Turing machine M, we assume, w.l.o.g., that M has only one halting state h, represented by a binary string "11 . . . 1" of an even length $2k$; and assume each of the other states of M is represented by a binary string of length $2k$ in which the number of '0' is the same as that of '1'. By the approach proposed in [6], one can construct an opinion diffusion $\mathcal{OD}_M$ (with threshold $1/2$) to simulates M. W.l.o.g., suppose $(x_1, y_1), \ldots, (x_{2k}, y_{2k})$ are the dual pairs in $\mathcal{OD}_M$ that encode the current state of M. Keep in mind that, in the binary representation of each state, '0' is encoded by the dual pair $(0, 1)$, and '1' by $(1, 0)$. Let n be the number of vertices in $\mathcal{OD}_M$, and let m be the least integer that is not less than $n/2$. Now we define a new opinion diffusion $\mathcal{OD}'_M$, obtained from $\mathcal{OD}_M$ by

1. adding fresh vertices $v_1, \ldots, v_{2m}$ into $\mathcal{OD}'_M$, setting the initial opinion of $v_1, \ldots, v_m$ to be 1, and setting the initial opinion of $v_{m+1}, \ldots, v_{2m}$ to be 0;
2. adding fresh vertices $u_1, \ldots, u_{4m+1}$ into $\mathcal{OD}'_M$, and setting the initial opinion of $u_1, \ldots, u_{4m+1}$ to be 1.
3. for each $i \in \{1, \ldots, 2m\}$, adding edges $(x_1, v_i), \ldots, (x_{2k}, v_i)$ into $\mathcal{OD}'_M$.
4. for each vertex v in $\mathcal{OD}_M$, adding edges $(v_1, v), \ldots, (v_{2m}, v)$ into $\mathcal{OD}'_M$.

Intuitively, vertices $v_1, \ldots, v_{2m}$ and related edges are used to check whether M halts; if true, then change their opinion to 1; by applying the diffusion, the opinion of any other vertex in $\mathcal{OD}'_M$ will be changed to 1. Vertices $u_1, \ldots, u_{4m+1}$ are introduced to assure that more than one half of vertices in O'_M hold the initial opinion 1.

Clearly, the size of $\mathcal{OD}'_M$ is still polynomial w.r.t. the size of M. It is also not difficult to see that M halts iff $\mathcal{OD}'_M$ is majority dominating. These then prove the correctness of the reduction.

□

The PSPACE-hardness indicates that, unless PTIME = PSPACE, it is impossible to identify features of a directed graph that characterize majority dominating opinion diffusion and are tractable to verify. The result sets the tone for our exploration of the majority to unanimity problem in which we aim for intuitively appealing and simple features of a directed graph that ensure majority dominating over some typical instances of opinion diffusion, and necessary (but not sufficient) features over all instances of opinion diffusion.

3 Controlling Set and Opinion Diffusion

In this section, we first introduce a graph-theoretic notion called *controlling set*, then we derive some of its important properties, and finally, we demonstrate its capability of fully characterizing the behaviour of opinion diffusion, which paves the way for obtaining network features that are sufficient or necessary for the majority dominating property of opinion diffusion. The notion of controlling set is not entirely new. For instance, the main idea behind it has been discussed in [9] which gives rise to the notions of *winning and veto coalitions* and *dependency sequence*. Apart from slight variations in the problem setting, the notions are essentially the same as that of controlling set. The difference is that, in this paper, we give a more comprehensive and systematic treatment of the properties of the notion, the intuitions behind it, and special classes of it. Also, while [9] uses the notion to form conditions for the stabilization of an opinion diffusion, we use it for the majority dominating property.

Lies in the heart of an opinion diffusion is the back and forth of groups of individuals gaining control of the opinions of each other. With controlling set, we intend to capture the controlling relationship between groups of individuals in terms of the structure of the underlying network. More precisely, given two sets of individuals C and S, the controlling relationship between C and S should imply the following: if, at some time point, individuals of C all have 1 (0) as their opinions, then, at some later time point, the individuals of C and S all have 1 (resp. 0) as their opinions. Essentially, the definition of controlling set is a reinterpretation of the update rule through graph-theoretic notions. For convenience, unless stated otherwise, in the remaining of the paper, the letters C, S, R denote a subset of $\mathcal{N}$. We define that C is a *level 1 controlling set* for S with respect to a threshold T, iff for any individual i of S, the proportion of incoming edges from C is greater than T when i is not in C and is no less than $1 - T$ when i is in C. Generalizing to arbitrary levels, C is a *level k controlling set* for S, iff there is a sequence of $k + 1$ sets that starts with C and ends with S such that any set in the sequence is a level 1 controlling set for the next one in the sequence. We say C *controls* S if there is a k such that C is a level k controlling set for S.

Definition 2. *The set of* level k controlling sets *for S with respect to a threshold T, denoted as $\mathcal{C}^k_T(S)$, is such that $C \in \mathcal{C}^1_T(S)$ iff*

1. $d_C(i)/d(i) > T$ for $i \in S \setminus C$, and

2. $d_C(i)/d(i) \geqslant 1 - T$ for $i \in S \cap C$.

For $k \geqslant 2$, $C \in \mathcal{C}^k_T(S)$ iff there are $C_0, C_1, \ldots, C_k$ such that $C_0 = C$, $C_k = S$ and $C_i \in \mathcal{C}^1_T(C_{i+1})$ for $0 \leqslant i \leqslant k - 1$.

Condition 1 takes care of the case where i of S is not in C, which means i may hold the opposite opinion than the unified opinion of C. Thus to ensure it will change its opinion to the unified opinion of C, the proportion of its influencers in C has to be higher than T. Condition 2 takes care of the case where i is in C. Since the notion of controlling set is intended solely for situations when all individuals of C have the same opinion, we only need to ensure i does not change its opinion which means the proportion of its influencers outside C has to be less than T, or equivalently, the proportion of its influencers in C is greater than or equal to $1 - T$.

Let's illustrate the definition with the opinion diffusion in Figure 1. Taking individual 4 as an example, since

$$d(4) = |I^-(4)| = |\{1, 2, 3, 4, 5\}| = 5$$

and

$$d_{\{1,2,3\}}(4) = |I^-_{\{1,2,3\}}(4)| = |I^-(4) \cap \{1, 2, 3\}| = |\{1, 2, 3\}| = 3$$

we have $d_{\{1,2,3\}}(4)/d(4) = 3/5 > 1/2$ which means $\{1,2,3\} \in \mathcal{C}^1_{1/2}(\{4\})$. For other individuals, since $d_{\{1,2,3\}}(3)/d(3) = 3/4 > 1/2$, $d_{\{1,2,3\}}(5)/d(5) = 3/4 > 1/2$, and $d_{\{1,2,3\}}(6)/d(6) = 2/3 > 1/2$, we have $\{1,2,3\} \in \mathcal{C}^1_{1/2}(\{3,4,5,6\})$. One can also verify that $\{3,4,5,6\} \in \mathcal{C}^1_{1/2}(\{1,2,3,4,5,6,7\})$ which implies $\{1,2,3\} \in \mathcal{C}^2_{1/2}(\{1,2,3,4,5,6,7\})$, that is $\{1,2,3\}$ is a level 2 controlling set for $\mathcal{N}$.

We first derive some basic properties of controlling set. Summarised in Lemma 1, the relationship induced by controlling set is "monotonic" that if C controls S with respect to a threshold T, then this also holds for any subset of S (point 1), any superset of C (point 2), and any smaller threshold than T (point 3). It is "additive" that C is a level k controlling set for S which is a level l controlling set for R implies C is a level $k+l$ controlling set for R (point 4). Finally, it is "exclusive" that non-overlapping sets cannot control the same set (point 5).

Lemma 1. *Let C, R, S be sets of individuals. Then the following holds:*

1. *if $C \in \mathcal{C}^k_T(S)$ and $R \subset S$, then $C \in \mathcal{C}^k_T(R)$;*
2. *if $C \in \mathcal{C}^k_T(S)$ and $C \subset R$, then $R \in \mathcal{C}^k_T(S)$;*
3. *if $T > T'$ and $C \in \mathcal{C}^k_T(S)$, then $C \in \mathcal{C}^k_{T'}(S)$;*
4. *if $C \in \mathcal{C}^k_T(S)$ and $S \in \mathcal{C}^l_T(R)$, then $C \in \mathcal{C}^{k+l}_T(R)$;*
5. *if $C, R \in \mathcal{C}^k_T(S)$, then $C \cap R \neq \emptyset$;*

Proof. We only provide the proof for point 5 as the rest follow immediately from Definition 2.

Point 5: Let $A, B \subseteq \mathcal{N}$ and $A \cap B = \emptyset$. For all $i \in \mathcal{N}$, $A \in \mathcal{C}^1_T(i)$ implies $d_A(i)/d(i) = |I^-(i) \cap A|/d(i) > T$. It follows from $|I^-(i) \cap A|/d(i) > T$ that $|I^-(i) \setminus (I^-(i) \cap A)|/d(i) < 1 - T$. Then as $(I^-(i) \cap B) \subseteq (I^-(i) \setminus (I^-(i) \cap A))$ follows from $A \cap B = \emptyset$, we have $d_B(i)/d(i) = |I^-(i) \cap B|/d(i) \leqslant |I^-(i) \setminus (I^-(i) \cap A)|/d(i) < 1 - T$ which means $B \notin \mathcal{C}^1_T(i)$. Hence, for all $A, B \subseteq \mathcal{N}$, $A \cap B = \emptyset$ implies there is no $i \in \mathcal{N}$ such that $A \in \mathcal{C}^1_T(i)$ and $B \in \mathcal{C}^1_T(i)$.

Suppose $C, R \in \mathcal{C}^k_T(S)$. Then there are sequences $C_0, C_1, \ldots, C_k$ and $R_0, R_1, \ldots, R_k$ such that $C_0 = C$, $R_0 = R$, $C_k = R_k = S$, and $C_i \in \mathcal{C}^1_T(C_{i+1})$, $R_i \in \mathcal{C}^1_T(C_{i+1})$ for $0 \leqslant i \leqslant k-1$. Let's assume $C \cap R = \emptyset$. Then according to the above result, $R_i \cap C_i = \emptyset$ for $1 \leqslant i \leqslant k$, which is a contradiction. Hence $C \cap R \neq \emptyset$.

□

Let C be a level k controlling set for S. Inspired by the monotonicity of the controlling relationship, it is natural to ask if S is the maximal set that C controls.[4] This leads to the notion of *maximal controlling sets*.

Definition 3. *The set of* maximal level k controlling sets *for S with respect to T, denoted as $\mathcal{MC}^k_T(S)$, is such that $C \in \mathcal{MC}^k_T(S)$ iff $C \in \mathcal{C}^k_T(S)$ and there is no $R \supset S$ such that $C \in \mathcal{C}^k_T(R)$.*

While there can be many maximal level k controlling sets for a given set, every set can be a maximal level k controlling set for exactly one set. In Figure 1, $\{1,2,3\}$ is an maximal level 1 controlling set for $\{3,4,5,6\}$. Note that any set that controls $\mathcal{N}$ is trivially a maximal controlling set for $\mathcal{N}$ as there is no set larger than $\mathcal{N}$. With this notion of maximality, we can derive that the controlling relationship is "complementary" that for any individual $i \in \mathcal{N}$, if a set C does not control it, the complement of C (with respect to $\mathcal{N}$) does.

[4] It is also natural to ask if C is the minimal set that controls S. The notion however is irrelevant to the main results of this paper.

Lemma 2. *Let $C \subseteq \mathcal{N}$ and $i \in \mathcal{N}$. If $C \notin \mathcal{C}^k_T(\{i\})$, then $\mathcal{N} \setminus C \in \mathcal{C}^k_T(\{i\})$.*

Proof. Let $S \subseteq \mathcal{N}$. We first show that, if $S \notin \mathcal{C}^1_T(\{i\})$, then $\mathcal{N} \setminus S \in \mathcal{C}^1_T(\{i\})$. There are two cases.

Case 1, $i \in S$: Then $i \notin \mathcal{N} \setminus S$. Also, by Definition 2, $d_S(i)/d(i) = |I^-(i) \cap S|/d(i) < 1 - T$ which implies $|I^-(i) \setminus (I^-(i) \cap S)|/d(i) > T$. Then since $I^-(i) \setminus (I^-(i) \cap S) = I^-(i) \cap (\mathcal{N} \setminus S)$, we have $|I^-(i) \cap (\mathcal{N} \setminus S)|/d(i) = d_{\mathcal{N} \setminus S}/d(i) > T$. Hence, by Definition 2, $\mathcal{N} \setminus S \in \mathcal{C}^1_T(\{i\})$.

Case 2, $i \notin S$: Then $i \in \mathcal{N} \setminus S$. Also, by Definition 2, $d_S(i)/d(i) = |I^-(i) \cap S|/d(i) \leqslant T$ which implies $|I^-(i) \setminus (I^-(i) \cap S)|/d(i) \geqslant 1 - T$. Then since $I^-(i) \setminus (I^-(i) \cap S) = I^-(i) \cap (\mathcal{N} \setminus S)$, we have $|I^-(i) \cap (\mathcal{N} \setminus S)|/d(i) = d_{\mathcal{N} \setminus S}/d(i) \geqslant 1 - T$. Hence, by Definition 2, $\mathcal{N} \setminus S \in \mathcal{C}^1_T(\{i\})$.

Thus, we have proved the above claim.

Now suppose $C \notin \mathcal{C}^k_T(\{i\})$. Let $C \in \mathcal{MC}^k_T(\{C_k\})$ (Note that such C_k always exists.). It follows from $C \in \mathcal{MC}^k_T(\{C_k\})$ that there is a sequence $C_0, C_1, \ldots, C_k$ such that $C = C_0$ and $C_i \in \mathcal{MC}^1_T(C_{i+1})$ for $0 \leqslant i \leqslant k-1$. Thus $i \notin C_k$ and by Definition 3 $C_{i-1} \notin \mathcal{C}^1_T(\{j\})$ for all $j \in \mathcal{N} \setminus C_i$ and $0 \leqslant i \leqslant k-1$. It then follows from the above claim that $\mathcal{N} \setminus C_i \in \mathcal{MC}^1_T(\mathcal{N} \setminus C_{i+1})$ for $0 \leqslant i \leqslant k-1$ which implies $\mathcal{N} \setminus C \in \mathcal{C}^k_T(\mathcal{N} \setminus C_k)$. Since $i \in \mathcal{N} \setminus C_k$, we have $\mathcal{N} \setminus C \in \mathcal{C}^k_T(\{i\})$.

□

An easy consequence of the complementary property is that S is a maximal level k controlling set for R iff their complements also form such an relationship.

Corollary 1. *Let $S, R \subseteq \mathcal{N}$. Then $S \in \mathcal{MC}^k_T(R)$ iff $\mathcal{N} \setminus S \in \mathcal{MC}^k_T(\mathcal{N} \setminus R)$ for all k.*

Therefore, for the opinion diffusion in Figure 1, it follows from $\{1,2,3\} \in \mathcal{MC}^1_{1/2}(\{3,4,5,6\})$ that $\{4,5,6\} \in \mathcal{C}^1_{1/2}(\{1,2,7\})$.

An interesting special case of the controlling relationship is for a set to control itself. With the notion of maximal controlling set, we can capture such *self controlling sets* more concisely.

Definition 4. *Let $S \subseteq \mathcal{N}$. Then S is a* self controlling set *if $S \in \mathcal{MC}^k(S)$ for some k. Furthermore, S is* a resistant set *if $S \in \mathcal{MC}^1_T(S)$.*

Note that $\mathcal{N}$ is trivially a self controlling as well as a resistant set. A crucial property of a self controlling set is that it cannot control $\mathcal{N}$, unless it is itself $\mathcal{N}$.

Lemma 3. *Let $S \subset \mathcal{N}$. If S is a self controlling set, then $S \notin \mathcal{C}^k_T(\mathcal{N})$ for all k.*

Proof. Suppose S is a self controlling set. Then $S \in \mathcal{MC}^t_T(S)$ for some t. Let's assume $S \in \mathcal{C}^k_T(\mathcal{N})$ for some k. Then there are two cases.

Case 1, $k \leqslant t$: Since $\mathcal{N} \in \mathcal{C}^l_T(\mathcal{N})$ for all l, it follows from $S \in \mathcal{C}^k_T(\mathcal{N})$ and Lemma 1 (point 4) that $S \in \mathcal{C}^t_T(\mathcal{N})$. But $S \in \mathcal{MC}^t_T(S)$ and $S \subset \mathcal{N}$, so we have a contradiction.

Case 2, $k > t$: Then $S \in \mathcal{C}^k_T(\mathcal{N})$ implies there is a sequence $S_0, S_1, \ldots, S_t, \ldots S_k$ such that $S = S_0$, $S_k = \mathcal{N}$, and $S_i \in \mathcal{C}^1_T(S_{i+1})$ for $0 \leqslant i \leqslant k-1$. Since $S \in \mathcal{MC}^t_T(S)$, we have $S_t \subseteq S$. It then follows from Lemma 1 (point 2) that $S \in \mathcal{C}^{k-t}_T(\mathcal{N})$. If $k - t \leqslant t$, then we have a contradiction as in Case 1. Otherwise, we can keep repeating the above steps to obtain $S \in \mathcal{C}_T^l(\mathcal{N})$ for some $l \leqslant t$.

Since both cases lead to contradiction, we conclude that $S \notin \mathcal{C}_T^k(\mathcal{N})$ for all k.

□

Consequently, a self controlling set is able to "block" all other sets from controlling $\mathcal{N}$, except for $\mathcal{N}$ itself. That is, unless R is $\mathcal{N}$, R cannot control $\mathcal{N}$ whenever R is a maximal controlling set for a self controlling set. The blocking behaviour turns out to be the cornerstone for several main results in the paper.

Lemma 4. *Let $R, S \subset \mathcal{N}$. If S is a self controlling set and $R \in \mathcal{MC}_T^t(S)$ for some t, then $R \notin \mathcal{C}_T^k(\mathcal{N})$ for all k.*

Now let's turns our attention to the most "resistant" kind of self controlling set—the resistant set. We can show that a resistant set can never be controlled by individuals outside the resistant set.

Lemma 5. *Let $S \subset \mathcal{N}$. If S is a resistant set, then $\mathcal{N} \setminus S \notin \mathcal{C}_T^k(S)$ for all k.*

Proof. Suppose S is a resistant set. Thus we have $S \in \mathcal{MC}_T^1(S)$. It then follows from Lemma 1 (point 4) and Definition 3 that $S \in \mathcal{MC}_T^k(S)$ for all k. Since $S \in \mathcal{MC}_T^k(S)$ for all k, it follows from Corolloary 1 that $\mathcal{N} \setminus S \in \mathcal{MC}_T^k(\mathcal{N} \setminus S)$ for all k which means $\mathcal{N} \setminus S \notin \mathcal{C}_T^k(S)$ for all k.

□

So far we have defined and explored controlling sets purely as a graph-theoretic notion. It's time to see how it relates and characterizes opinion diffusion. Actually, for an opinion diffusion $\mathcal{OD} = (\mathcal{G}, T, \mathbf{O}^0)$, any pair of sets in the sequence $\mathbf{1}_{\mathbf{O}^0}, \mathbf{1}_{\mathbf{O}^1}, \ldots, \mathbf{1}_{\mathbf{O}^t}$ form a maximal controlling set relationship. Recall that $\mathbf{1}_{\mathbf{O}^i}$ uniquely determines the opinion profile $\mathbf{O}^i$.

Proposition 2. *For an opinion diffusion $\mathcal{OD} = (\mathcal{G}, T, \mathbf{O}^0)$,*

$$\mathbf{1}_{\mathbf{O}^k} \in \mathcal{MC}_T^{t-k}(\mathbf{1}_{\mathbf{O}^t})$$

for all $t > k$.

Proof. By Lemma 1 (point 4), it suffices to show $\mathbf{1}_{\mathbf{O}^k} \in \mathcal{MC}_T^1(\mathbf{1}_{\mathbf{O}^{k+1}})$. It is clear from Definition 2 and the update rule that $\mathbf{1}_{\mathbf{O}^k} \in \mathcal{C}_T^1(\mathbf{1}_{\mathbf{O}^{k+1}})$. Suppose $i \notin \mathbf{1}_{\mathbf{O}^{k+1}}$, then $O^{k+1} = 0$. Then it follows from the update rule that either $d_{\mathbf{1}_{\mathbf{O}^k}}(i)/d(i) \leqslant T$ when $i \notin \mathbf{1}_{\mathbf{O}^k}$ or $d_{\mathbf{1}_{\mathbf{O}^k}}(i)/d(i) < 1 - T$ when $i \in \mathbf{1}_{\mathbf{O}^k}$. Then we have by Definition 2 that $\mathbf{1}_{\mathbf{O}^k} \in \mathcal{C}_T^1(\{i\})$ which means $\mathbf{1}_{\mathbf{O}^k} \in \mathcal{MC}_T^1(\mathbf{1}_{\mathbf{O}^{k+1}})$.

□

An opinion diffusion stabilizes if it gets into an opinion profile $\mathbf{O}^t$ and sticks with it. According to Proposition 2 this happens only if $\mathbf{1}_{\mathbf{O}^t}$ is a resistant set. Furthermore, if the resistant set $\mathbf{1}_{\mathbf{O}^t}$ is $\mathcal{N}$, then the opinion diffusion also converges at t. That is, an opinion diffusion $\mathcal{OD} = (\mathcal{G}, T, \mathbf{O}^0)$ stabilizes at t iff there is $S \subseteq \mathcal{N}$ (denoting the set $\mathbf{1}_{\mathbf{O}^t}$) such that S is a resistant set and either $\mathbf{1}_{\mathbf{O}^0}$ or $\mathbf{0}_{\mathbf{O}^0}$ controls S; and the opinion diffusion converges at t iff either $\mathbf{1}_{\mathbf{O}^0}$ or $\mathbf{0}_{\mathbf{O}^0}$ controls $\mathcal{N}$. Generalizing to arbitrary initial profiles, we obtain the following conditions for the stabilization and convergence of opinion diffusion.

Proposition 3. *$\mathcal{OD} = (\mathcal{G}, T, \mathbf{O}^0)$ stabilizes for all $\mathbf{O}^0$ iff for all $C \subseteq \mathcal{N}$, there is $S \subseteq \mathcal{N}$ such that S is a resistant set and either $C \in \mathcal{MC}_T^k(S)$ or $\mathcal{N} \setminus C \in \mathcal{MC}_T^k(S)$ for some k*

$\mathcal{OD} = (\mathcal{G}, T, \mathbf{O}^0)$ converges for all $\mathbf{O}^0$ iff for all $C \subseteq \mathcal{N}$, either $C \in \mathcal{MC}_T^k(\mathcal{N})$ or $\mathcal{N} \setminus C \in \mathcal{MC}_T^k(\mathcal{N})$ for some k.

Note that the set C in Proposition 3 is intended to denote the set $\mathbf{1}_{\mathbf{O}^0}$.

We have completed our exploration of controlling set. Apart from clarifying various aspects of the opinion diffusion process, it allows us to investigate the process through static and pure graph-theoretic terms without the need to deal with the iterated and dynamical nature of opinion diffusion.

4 Guaranteeing Majority Dominating

In this section, we identify features of directed graphs that lead to majority dominating opinion diffusion. Due to the PSPACE hardness, we cannot expect to obtain sufficient and necessary features that are easy to verify and generalize to all instances of opinion diffusion. Our goal is instead to identify intuitively appealing and simple majority dominating features for typical instances of opinion diffusion. We first characterize majority dominating opinion diffusions through controlling set and then infer their implication to the indegree and/or neighbourhood of the vertices as well as the existence of some controlling relationships. As a complement to the majority to unanimity problem, we also discuss opinion diffusion for which the initial opinion profile is without a majority opinion.

Following the characterization of convergence, to ensure majority dominating for all initial profiles, we need all sets of individuals with cardinality greater than $n \cdot T$ (i.e., denoting the initial majority opinion) to control $\mathcal{N}$.

Proposition 4. *$\mathcal{OD} = (\mathcal{G}, T, \mathbf{O}^0)$ is majority dominating for all $\mathbf{O}^0$ iff for all $S \subseteq \mathcal{N}$ with $|S| > n \cdot T$, $S \in \mathcal{C}_T^k(\mathcal{N})$ for some k.*

According to the characterisation, to ensure majority dominating, it suffices to identity features that guarantee any set of individuals with cardinality greater than $n \cdot T$ controls $\mathcal{N}$. We can derive an equivalent formulation for such controlling relationships which reveals two important intuitions for majority dominating. That is, firstly, any set of individuals with cardinality greater than $n \cdot T$ and those maximally controlled by one such set cannot be a self controlling set and, secondly, any set of individuals with cardinality less than $n \cdot (1 - T)$ cannot control a set with cardinality greater than $n \cdot T$.

Proposition 5. *For all $S \subseteq \mathcal{N}$ with $|S| > n \cdot T$, $S \in \mathcal{C}_T^k(\mathcal{N})$ for some k iff*

1. *if $|S| > n \cdot T$ or there is R such that $|R| > n \cdot T$ and $R \in \mathcal{MC}^k(S)$ for some k, then S is not a self controlling set; and*
2. *if $|R| < n \cdot (1 - T)$ and $|S| > n \cdot T$, then R does not control S.*

Proof. $\Rightarrow$: Suppose for all S with $|S| > n \cdot T$, S controls $\mathcal{N}$. For condition 1, it follow from Lemma 4 (point 2) that we have a contradiction if S is a self controlling set. For condition 2, assume to the contrary that R controls S. Since $|S| > n \cdot T$, S controls $\mathcal{N}$. Then as R controls S, it follows from Lemma 1 (point 4) that R controls $\mathcal{N}$. Since R controls $\mathcal{N}$, it follows from Lemma 1 (point 5) that $\mathcal{N} \setminus R$ does not control $\mathcal{N}$. As $|\mathcal{N} \setminus R| = n - |R| > n - n \cdot (1 - T) = n \cdot T$, it follows from the premises that $\mathcal{N} \setminus R$ controls $\mathcal{N}$. So we have a contradiction.

$\Leftarrow$: We prove the contrapositive, that is if there is S such that $|S| > n \cdot T$ and S does not control $\mathcal{N}$, then either condition 1 or condition 2 is violated. Since there is a finite number of individuals, the sequence $S, C_1, C_2, \ldots$ where $S \in \mathcal{MC}^1(C_1)$ and $C_i \in \mathcal{MC}^1(C_{i+1})$ for all $i \geqslant 1$ either runs into a cycle or end with some $C_k = \emptyset$ which does not control any individual. In the former case, since S does not control $\mathcal{N}$, S controls some $R \subset \mathcal{N}$ that is a self controlling set, which violates condition 1. In the latter case, since $S \in \mathcal{MC}^k(\emptyset)$ and $\mathcal{N} \setminus \emptyset = \mathcal{N}$, it follows from Corollary 1 that $\mathcal{N} \setminus S \in \mathcal{MC}^k(\mathcal{N})$. Then since $|S| > n \cdot T$ implies $|\mathcal{N}/S| < n \cdot (1-T)$, condition 2 is violated.

□

For condition 1, according to Lemma 4, self controlling sets have the effect of blocking control over $\mathcal{N}$. Therefore, we cannot bear to have too many self controlling sets if the goal is to gain control over $\mathcal{N}$. For condition 2, it shows that there cannot be any switch of the majority opinion if $(\mathcal{G}, T, \mathbf{O}^0)$ is majority dominating for all $\mathbf{O}^0$. That is, if the initial majority opinion is 1, then any of the subsequent opinion profiles cannot have a majority opinion of 0.

Next, we give an implication of the characterising controlling relationship, which reveals the lower bound on the size of some "neighbourhood." The closer T is to $1/2$, the more informative the condition is.

Proposition 6. *If for all $S \subseteq \mathcal{N}$ with $|S| > n \cdot T$, $S \in \mathcal{C}_T^k(\mathcal{N})$ for some k, then*

$$|I^-(S) \cup I^+(S) \cup I^-(I^+(S))| > n \cdot T$$

for all S such that $i, j \in S$ implies $I^+(i) \cap I^+(j) = \emptyset$ and $I^-(i) \cap I^-(j) = \emptyset$, and $|S| > 2n \cdot T - n + 1$ for when $n \cdot T$ is an integer, and $|S| > 2\lceil n \cdot T \rceil - n$ otherwise.

Proof. Suppose S controls $\mathcal{N}$ for all S with $|S| > n \cdot T$. Then there is R such that $S \in \mathcal{C}^1(R)$ and R controls $\mathcal{N}$. It follows from Proposition 5 (condition 1) that $d(i) > 1/(1-T)$ for all $i \in \mathcal{N}$. Note that if $d(i) \leqslant 1/(1-T)$, then $\{i\}$ is a resistant set. Assume to the contrary that $|I^-(S) \cup I^+(S) \cup I^+(I^+(S))| \leqslant n \cdot T$. Suppose $n \cdot T$ is an integer. Let $A \subseteq \mathcal{N}$ be such that $|A| = n \cdot T + 1$ and $I^-(S) \cup I^+(S) \cup I^+(I^+(S)) \subseteq A$. We will show $A \setminus S \in \mathcal{C}^1(R)$. Let $j \in R$. There are three cases.

Case 1, $j \notin I^+(S)$: then $d_{A \setminus S}(j) = d_A(j)$ which means $A \setminus S \in \mathcal{C}^1(\{j\})$.

Case 2, $j \in I^+(S) \setminus S$: Then it follows from $I^-(I^+(S)) \subseteq A$ that $d_A(j) = d(j)$. Then since, for all $i, j \in S$, $I^+(i) \cap I^+(j) = \emptyset$, we have $d_{A \setminus S}(j) = d_A(j) - 1$. It then follows from $d(i) > 1/(1-T)$ for all $i \in \mathcal{N}$ that $d_{A \setminus S}(j)/d(j) = (d_A(j) - 1)/d(j) = 1 - 1/d(j) > 1 - 1/(1/(1-T)) = T$. Hence $A \setminus S \in \mathcal{C}^1(\{j\})$.

Case 3, $j \in S$: Then it follows from $I^-(S) \subseteq A$ that $d_A(j) = d(j)$. Then since, for all $i, j \in S$, $I^-(i) \cap I^-(j) = \emptyset$, we have $d_{A \setminus S}(j) = d_A(j) - 1$. It then follows from $d(i) > 1/(1-T)$ for all $i \in \mathcal{N}$ that $d_{A \setminus S}(j)/d(j) = (d_A(j) - 1)/d(j) = 1 - 1/d(j) > 1 - 1/(1/(1-T)) = T$. Hence $A \setminus S \in \mathcal{C}^1(\{j\})$.

Therefore, $A \setminus S \in \mathcal{C}^1(R)$ and it follows from Lemma 1 (point 4) that $A \setminus S$ controls $\mathcal{N}$. Since $|S| > 2n \cdot T - n + 1$, $|A \setminus S| < n \cdot T + 1 - (2n \cdot T - n + 1) = n \cdot (1-T)$. Then according to Proposition 5 (condition 2), we have a contradiction.

The proof is similar for when $n \cdot T$ is not an integer.

□

The condition states that for any set S with certain lower bound of cardinality where all its individuals do not influence or are influenced by the same individual, its close neighbourhood occupies a majority of vertices. The intuition for the lower bound of cardinality is that (for the case when $n \cdot T$ is an integer, and the other case is similar) it is the number of vertices that needs to be taken away to turn a set with a majority number (i.e., $n \cdot T + 1$) of vertices into one with a minority number (i.e., $n \cdot (1-T)$) of vertices. The close neighbourhood is formed to make sure the removal of all vertices in S does not affect what the neighbourhood controls when the size of the neighbourhood is less than $n \cdot T$. Note that as T gets closer to $1/2$, the lower bound of cardinality gets smaller. Hence we have an indication of the size of the close neighbourhood of a relatively smaller set of vertices. If T is exactly $1/2$ and n is odd, then the lower bound is 1. Thus we can derive that the close neighbourhood of a single vertex already occupies the majority of vertices. This means, for this threshold, the directed graphs that guarantee majority dominating opinion diffusion have to be very dense.

Furthermore, we characterize majority dominating opinion diffusion that stabilizes at time 1, because, in many real-time applications, the rapid convergence of an opinion diffusion is imperative. As a particular case for Proposition 4, we have that opinion diffusion is majority dominating and stabilizes at time 1 for all initial profiles iff for all $S \subseteq \mathcal{N}$ with cardinality greater than $n \cdot T$ it is a level 1 controlling set for $\mathcal{N}$. We can show that, for this to happen, the indegree of all vertices has to be greater than $n - 1/(1-T)$ when $n \cdot T$ is an integer and otherwise greater than $(n - \lceil n \cdot T \rceil)/(1-T)$.

Proposition 7. *$\mathcal{OD} = (\mathcal{G}, T, \mathbf{O}^0)$ is majority dominating for all $\mathbf{O}^0$ and stabilizes at time 1, iff*

$$d(i) > n - 1/(1-T)$$

when $n \cdot T$ is an integer, otherwise

$$d(i) > (n - \lceil n \cdot T \rceil)/(1-T).$$

for all $i \in \mathcal{N}$.

Proof. $\Rightarrow$: Suppose $S \in \mathcal{C}^1(\mathcal{N})$ for all S with $|S| > n \cdot T$. That is, if $i \in \mathcal{N}$ and $|S| > n \cdot T$ then $d_S(i)/d(i) > T$ when $i \in \mathcal{N} \setminus S$ and $d_S(i)/d(i) \geqslant 1 - T$ otherwise.

Let $i \in \mathcal{N}$. If $d(i) < n \cdot (1-T)$, then there is $S \subset \mathcal{N}$ such that $S \cap I^-(i) = \emptyset$ which means $d_S(i)/d(i) = 0$. Hence we have $d(i) \geqslant n \cdot (1-T)$. There are two cases.

Case 1, $n \cdot T$ is an integer: Then there is $S \subset \mathcal{N}$ such that $|S| = n \cdot T + 1$, $\mathcal{N} \setminus I^-(i) \subset S$, and $i \in \mathcal{N} \setminus S$. Thus $d_S(i) = |S| - |\mathcal{N} \setminus I^-(i)| = n \cdot T + 1 - n + d(i)$. Since $d_S(i)/d(i) > T$, $(n \cdot T + 1 - n + d(i))/d(i) > T$ which implies $d(i) > n - 1/(1-T)$.

Case 2, $n \cdot T$ is not an integer: Then there is $S \subset \mathcal{N}$ such that $|S| = \lceil n \cdot T \rceil$, $\mathcal{N} \setminus I^-(i) \subset S$, and $i \in \mathcal{N} \setminus S$. Thus $d_S(i) = |S| - |\mathcal{N} \setminus I^-(i)| = \lceil n \cdot T \rceil - n + d(i)$. Since $d_S(i)/d(i) > T$, $(\lceil n \cdot T \rceil - n + d(i))/d(i) > T$ which implies $d(i) > (n - \lceil n \cdot T \rceil)/(1-T)$.

$\Leftarrow$: Suppose, for all $i \in \mathcal{N}$, $d(i) > n - 1/(1-T)$ when $n \cdot (1-T)$ is an integer and $d(i) > (n - \lceil n \cdot T \rceil)/(1-T)$ otherwise. So there are two cases:

Case 1, $n \cdot T$ is an integer: Due to Lemma 1 (point 2), it suffices to show for all $S \subset \mathcal{N}$ with $|S| = n \cdot T + 1$, $S \in \mathcal{C}^1(\mathcal{N})$, that is $d_S(i)/d(i) > T$ if $i \in \mathcal{N} \setminus S$ and $d_S(i)/d(i) \geqslant 1 - T$ otherwise. Let $S \subset \mathcal{N}$, $|S| = n \cdot T + 1$, and $i \in \mathcal{N}$. Then since $d_S(i) \geqslant |S| - |\mathcal{N} \setminus I^-(i)| = n \cdot T + 1 - n + d(i)$, $d_S(i)/d(i) \geqslant (n \cdot T + 1 - n + d(i))/d(i) = (n \cdot T + 1 - n)/d(i) + 1 = 1 - (n - n \cdot T - 1)/d(i) > 1 - (n - n \cdot T - 1)/(n - 1/(1-T)) = T$.

Case 2, $n \cdot T$ is not an integer: Due to Lemma 1 (point 2), it suffices to show for all $S \subset \mathcal{N}$ with $|S| = \lceil n \cdot T \rceil$, $S \in \mathcal{C}^1(\mathcal{N})$. Let $S \subset \mathcal{N}$, $|S| = \lceil n \cdot T \rceil$, and $i \in \mathcal{N}$. Then since $d_S(i) \geqslant |S| - |\mathcal{N} \setminus I^-(i)| = \lceil n \cdot T \rceil - n + d(i)$, $d_S(i)/d(i) \geqslant (\lceil n \cdot T \rceil - n + d(i))/d(i) = (\lceil n \cdot T \rceil - n)/d(i) + 1 = 1 - (n - \lceil n \cdot T \rceil)/d(i) > 1 - (n - \lceil n \cdot T \rceil)/(n - \lceil n \cdot T \rceil)/(1-T) = T$.

□

Proposition 7 confirms that we can characterize the one-step convergence of majority dominating opinion diffusion through bounding the indegree of vertices. For illustration, let T be $1/2$. Then the indegree of an vertex has to be greater than $n-2$ if n is even and greater than $n-1$ if n is odd. In other words, an vertex can only tolerate one missing incoming edge if n is even and if n is odd, it cannot tolerate any missing incoming edge, that is the directed graph has to be complete.

So far our focus has been on lifting the initial majority opinion to the unanimous opinion. But what if there is no majority opinion to begin with? Majority dominating rules out the possibility of an initial minority opinion becoming the unanimous opinion, but nothing is said about those without a majority opinion. For the rest of this section, we look into such opinion diffusions.

Since, in an opinion diffusion, the individuals aggregate their opinions through majority voting, if there is no majority opinion to begin with, one of the most sensible thing to do is for them to hold on to their initial opinions. That is an opinion diffusion stabilizes and the stabilized opinion profile is the initial one, whenever there is no initial majority opinion. For the ease of presentation, we denote such property of opinion diffusion as *tie preserving*.

It is not hard to see that, an opinion diffusion is tie preserving iff it stabilizes at time 0. It then follows from Lemma 5 that ensuring tie preserving is the same as ensuring $\mathbf{1}_{\mathbf{O}^0}$ or $\mathbf{0}_{\mathbf{O}^0}$ is a resistant set (which prevents any change to the initial profile). Hence, we have the following characterization in which the cardinality constraint dictates that the set S (denoting $\mathbf{1}_{\mathbf{O}^0}$ or $\mathbf{0}_{\mathbf{O}^0}$) neither form a majority nor a minority of $\mathcal{N}$.

Proposition 8. *$\mathcal{OD} = (\mathcal{G}, T, \mathbf{O}^0)$ is tie preserving for all $\mathbf{O}^0$ iff S is a resistant set for all $S \subseteq \mathcal{N}$ with $n \cdot (1-T) \leqslant |S| \leqslant n \cdot T$.*

By taking advantage of the properties of resistant set we can also characterize tie preserving by bounding the indegree of vertices. That is the indegree of all vertices has to be either no more than $1/(1-T)$ or no less than $\lfloor n \cdot T \rfloor / T$.

Proposition 9. *S is a resistant set for all $S \subseteq \mathcal{N}$ with $n \cdot (1-T) \leqslant |S| \leqslant n \cdot T$ iff either $d(i) \leqslant 1/(1-T)$ or $d(i) \geqslant \lfloor n \cdot T \rfloor / T$ for all $i \in \mathcal{N}$.*

Proof. $\Leftarrow$: Suppose, for all $i \in \mathcal{N}$, either $d(i) \leqslant 1/(1-T)$ or $d(i) \geqslant \lfloor n \cdot T \rfloor / T$. We first show that for all S with $n \cdot (1-T) \leqslant |S| \leqslant n \cdot T$, $d_S(i)/d(i) \leqslant T$ for all $i \in \mathcal{N} \setminus S$. Let S be such that $n \cdot (1-T) \leqslant |S| \leqslant n \cdot T$, and $i \in \mathcal{N} \setminus S$. Then there are two cases.

Case 1, $d(i) \leqslant 1/(1-T)$: Then $d_S(i)/d(i) \leqslant (d(i)-1)/d(i) = 1 - 1/d(i) \leqslant 1 - 1/(1/(1-T)) = T$.

Case 2, $d(i) \geqslant \lfloor n \cdot T \rfloor / T$: Then, since $|S| \leqslant n \cdot T$, we have $d_S(i)/d(i) \leqslant \lfloor n \cdot T \rfloor / d(i) \leqslant \lfloor n \cdot T \rfloor / (\lfloor n \cdot T \rfloor / T) = T$.

Now let S be such that $n \cdot (1-T) \leqslant |S| \leqslant n \cdot T$. Then $n \cdot (1-T) \leqslant |\mathcal{N} \setminus S| \leqslant n \cdot T$. By the above result $d_{\mathcal{N} \setminus S}(i)/d(i) \leqslant T$ for all $i \in S$. Since $d(i) = d_{\mathcal{N} \setminus S}(i) + d_S(i)$, we have $d_S(i)/d(i) = (d(i) - d_{\mathcal{N} \setminus S}(i))/d(i) = 1 - d_{\mathcal{N} \setminus S}(i)/d(i) \geqslant 1 - T$. Moreover, by the above result, $d_S(i)/d(i) \leqslant T$ for all $i \in \mathcal{N} \setminus S$. Therefore S is a resistant set.

$\Rightarrow$: Suppose for all S with $n \cdot (1-T) \leqslant |S| \leqslant n \cdot T$, S is a resistant set. This means for all $i \in \mathcal{N} \setminus S$, $d_S(i)/d(i) \leqslant T$ for all such S. Let $i \in \mathcal{N}$. Then there are two cases.

Case 1, $d(i) \leqslant n \cdot T$: Then there is $S \subseteq \mathcal{N}$ such that $|S| = \lfloor n \cdot T \rfloor$, $i \notin S$, and $I^-(i) \setminus \{i\} \subseteq S$, which implies $d_S(i) = d(i) - 1$ and $d_S(i)/d(i) \leqslant T$. It follows from $d_S(i) = d(i) - 1$ and $d_S(i)/d(i) \leqslant T$ that $d(i) \leqslant 1/(1-T)$.

Case 2, $d(i) > n \cdot T$: Then there is S such that $|S| = \lfloor n \cdot T \rfloor$, $i \notin S$, and $S \subseteq I^-(i)$ which implies $d_S(i) = |S| = \lfloor n \cdot T \rfloor$ and $d_S(i)/d(i) \leqslant T$. It follows from $d_S(i) = \lfloor n \cdot T \rfloor$ and $d_S(i)/d(i) \leqslant T$ that $d(i) \geqslant \lfloor n \cdot T \rfloor / T$.

$\square$

For illustration, let the threshold be $1/2$. Then the indegree of all vertices has to be either no more than 2 or no less than n $(n-1)$ if n is odd (resp. even). In other words, for any individual, either almost no other individuals influence it or almost all the other individuals influence it.

5 Related Work

Traditionally, the process of information exchange among a network of individuals are studied in economics and social sciences [16] for which many formal models such as the DeGroot model [10] and the threshold model [15] are proposed. In artificial intelligence the process has been investigated from the standpoints of belief revision and merging [22, 23, 7, 8, 24], reasoning about knowledge and belief [17], and more intensively multiagent systems [1, 2, 4, 5, 6, 7, 8, 9, 11, 12, 14].

The current work is mostly inspired by the recent endeavours in multiagent systems on opinion diffusion and in particular those of [4, 9, 14]. [14] works with a version of opinion diffusion in which the individuals may apply different methods in aggregating opinions on multiple issues. Most significantly, they show that if the underlying network is without a cycle (excluding self-loop) and every individual's aggregation method satisfies a monotonicity property, then the opinion diffusion converges w.r.t. all initial opinion profiles. Furthermore, they show that in the case that the individuals follow the majority rule in aggregating opinions, then convergence can also be guaranteed with cycles in the underlying network, provided that the cycles are vertex-disjoint, and all their nodes have an even number of incoming edges. [4] extends the work by applying various forms of constraints on the possible opinion profiles. With a similar setting, [9] emphases on stabilisation. They propose to characterize opinion diffusion from a modal logic perspective and form several stabilization conditions in terms of properties of wining and veto coalitions that can be expressed in the modal logic of μ-calculus.

The current work also finds connections in discrete mathematics [3, 18, 20, 21]. These works assume a network of individuals formalized as an undirected graph, and tries to identify a subset of individuals that can force all individuals to have the same opinion as theirs in an opinion diffusion. They call such subsets of individuals *dynamic monopolies* which are the sets that control $\mathcal{N}$ in our terminology. Many efforts have been given to construct graphs which contain a small dynamic monopoly and to establish lower bound on the size of dynamic monopolies. For instance, [3] proves that for every n, there exist a graph with at least n vertices containing a dynamic monopoly of size 18.

The current work is most closely related to that of [18] which identifies graphs that ensure majority dominating opinion diffusion and they call such a graph *majority consensus computer*. Assuming undirected graphs and a fixed threshold of 1/2, they provide a variety of results including that the diameter of a majority consensus computer is less than four, has a trivial min-cut and a non-unique max-cut. Since we work with directed graphs and arbitrary threshold, some of our results generalize theirs (i.e., Proposition 6).

6 Conclusion

In this paper, we motivated, formalized and studied the majority to unanimity problem of opinion diffusion. We first gave a thorough investigation on the notion of controlling set which is proven to be useful in deriving network features that lead to various desirable behaviours of opinion diffusion. By exploiting the notion of controlling set, we articulated conditions for guaranteeing the majority dominating property through bounding indegree of vertices and the size of some neighbourhoods. For future work, we plan to generalize our results to the asynchronous mode of opinion update and multiple issues with constraints on the possible opinion profiles.

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REFERENCES

[1] Vincenzo Auletta, Ioannis Caragiannis, Diodato Ferraioli, Clemente Galdi, and Giuseppe Persiano, 'Minority becomes majority in social networks', in *Web and Internet Economics*, eds., Evangelos Markakis and Guido Schäfer, pp. 74–88, (2015).

[2] Vincenzo Auletta, Diodato Ferraioli, and Gianluigi Greco, 'Reasoning about consensus when opinions diffuse through majority dynamics', in *Proceedings of the Twenty-Seventh International Joint Conference on Artificial Intelligence, IJCAI-18*, pp. 49–55, (2018).

[3] Eli Berger, 'Dynamic monopolies of constant size', *Journal of Combinatorial Theory, Series B*, **83**(2), 191–200, (2001).

[4] Sirin Botan, Umberto Grandi, and Laurent Perrussel, 'Multi-issue opinion diffusion under constraints', in *Proceedings of the 18th International Conference on Autonomous Agents and MultiAgent Systems (AAMAS-2019)*, pp. 828–836, (2019).

[5] Robert Bredereck and Edith Elkind, 'Manipulating opinion diffusion in social networks', in *Proceedings of the 26th International Joint Conference on Artificial Intelligence*, pp. 894–900, (2017).

[6] Dmitry Chistikov, Grzegorz Lisowski, Michael S. Paterson, and Paolo Turrini, 'Convergence of opinion diffusion is pspace-complete', in *Proceedings of the 34th AAAI Conference (AAAI-2020)*, (2020).

[7] Laurence Cholvy, 'Diffusion of opinion and influence', in *Proceedings of the 10th International Conference on Scalable Uncertainty Management*, pp. 112–125, (2016).

[8] Laurence Cholvy, 'Opinion diffusion and influence: A logical approach', *International Journal of Approximate Reasoning*, **93**, 24 – 39, (2018).

[9] Zoé Christoff and Davide Grossi, 'Stability in binary opinion diffusion', in *Proceedings of the 6th International Conference on Logic, Rationality, and Interaction*, pp. 166–180, (2017).

[10] Morris H. DeGroot, 'Reaching a consensus', *Journal of the American Statistical Association*, **69**(345), 118–121, (1974).

[11] Piotr Faliszewski, Rica Gonen, Martin Koutecký, and Nimrod Talmon, 'Opinion diffusion and campaigning on society graphs', in *Proceedings of the 27th International Joint Conference on Artificial Intelligence*, pp. 219–225, (2018).

[12] Diodato Ferraioli and Carmine Ventre, 'Social pressure in opinion games', in *Proceedings of the 26th International Joint Conference on Artificial Intelligence (IJCAI-2017)*, pp. 3661–3667, (2017).

[13] Jr. John R. P. French, 'A formal theory of social power.', *Psychological review*, **63**, 181–94, (1956).

[14] Umberto Grandi, Emiliano Lorini, and Laurent Perrussel, 'Propositional opinion diffusion', in *Proceedings of the 2015 International Conference on Autonomous Agents and Multiagent Systems*, pp. 989–997, (2015).

[15] Mark Granovetter, 'Threshold models of collective behavior', *American Journal of Sociology*, **83**(6), 1420–1443, (1978).

[16] Matthew O. Jackson, *Social and Economic Networks*, Princeton University Press, 2008.

[17] Fenrong Liu, Jeremy Seligman, and Patrick Girard, 'Logical dynamics of belief change in the community', *Synthese*, **191**(11), 2403–2431, (2014).

[18] N. Mustafa and A. Pekec, 'Listen to your neighbors: How (not) to reach a consensus', *SIAM Journal on Discrete Mathematics*, **17**(4), 634–660, (2004).

[19] Christos H. Papadimitriou, *Computational complexity*, Addison-Wesley, 1994.

[20] D. Peleg, 'Size bounds for dynamic monopolies', *Discrete Applied Mathematics*, **86**(2), 263–273, (1998).

[21] David Peleg, 'Local majorities, coalitions and monopolies in graphs: a review', *Theoretical Computer Science*, **282**(2), 231–257, (2002).

[22] Nicolas Schwind, Katsumi Inoue, Gauvain Bourgne, Sébastien Konieczny, and Pierre Marquis, 'Belief revision games', in *Proceedings of the 29th AAAI Conference on Artificial Intelligence*, pp. 1590–1596, (2015).

[23] Nicolas Schwind, Katsumi Inoue, Gauvain Bourgne, Sébastien Konieczny, and Pierre Marquis, 'Is promoting beliefs useful to make them accepted in networks of agents?', in *Proceedings of the 25th International Joint Conference on Artificial Intelligence*, pp. 1237–1243, (2016).

[24] Nicolas Schwind and Pierre Marquis, 'On consensus in belief merging', in *Proceedings of the 32nd AAAI Conference on Artificial Intelligence*, pp. 1949–1956, (2018).