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Studying lack of information through type-2 fuzzy strong negation

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Abstract. Fuzzy systems are a kind of knowledge based system where information and knowledge are represented in the form of fuzzy statements. Modeling the negation of these statements plays an important role in fuzzy inference. As any other knowledge based system, fuzzy systems may be affected by a lack of information. In particular, in the case of type-2 fuzzy systems, the fuzzy set **1** ($\mathbf{1}(x) = 1$ for all $x \in [0, 1]$) represents a lack of information. Consequently, it is interesting to consider how this set (representing lack of information) is transformed through a negation.

The analysis in this paper takes place in the framework defined by **M**, the set of all functions from $[0, 1]$ to $[0, 1]$. These functions constitute the membership functions characterizing type-2 fuzzy sets. In particular we will concentrate on **L**, the set of normal and convex functions in **M**. In a first step, we will explore strong negations in **L** to obtain any possible image of the function **1** through them. The particular case of strong negations mapping **1** to **1** has been previously studied. In this work we will generalize the previous results by considering the main properties of those strong negations in **L** where the image of the function **1** is not this same function.

1 INTRODUCTION

Type-2 fuzzy sets (T2FSs) were introduced by L.A. Zadeh in 1975 ([27]) as an extension of the fuzzy sets (FS) also introduced by Zadeh in 1965 ([26]). We can also refer to these *original* fuzzy sets as type-1 fuzzy sets (T1FSs). These two types of sets differ in the ranges of their membership functions. While the membership functions of T1FSs take values in the interval $[0, 1]$, the values of the membership function of a T2FS are fuzzy sets in $[0, 1]$. That is, the degree of membership of an element a to a T2FS A, could be described by a label of the linguistic variable "TRUTH". In this way, a T2FS is determined by a membership function $\mu : X \to \mathbf{M}$, where $\mathbf{M} = [0, 1]^{[0,1]}$ is the set of the functions from $[0, 1]$ to $[0, 1]$ ($\mathbf{M} = \mathit{Map}([0, 1], [0, 1])$), see [16], [17], [23].

In general, FSs are well suited to work with uncertainty, but this extension of membership degrees from values in $[0, 1]$ to fuzzy sets in $[0, 1]$ provides T2FSs with an additional capability to model uncertainty ([4], [14], [16]). In the last decade, many researches have studied type-2 fuzzy sets, considering the theoretical aspects as well as its application to different areas ([7], [12], [24], [25], [28]).

It is obvious that Type-1 and Type-2 fuzzy sets are strongly related. Consequently, once introduced T2FSs, many operators, properties and results initially defined for T1FSs were adapted to T2FSs ([8], [11], [16], [17], [23]) by means of Zadeh's Extension Principle ([27]). This is the case of negation, a widely used operator that is involved, for example, in obtaining the complement of a set, and the dual of a t-norm or t-conorm operator. Some definitions of entropy and implications also involve the use of negations.

Fuzzy systems (either T1FSs or T2FSs) are a kind of knowledge based system where information and knowledge are represented in the form of fuzzy statements. Negation is an important tool when analyzing the potential contradiction, inconsistency or incoherence among pieces of knowledge in a fuzzy system. As an example, strong negations on Atanassov's intuitionistic fuzzy sets (A-IFSs) and on T2FSs have been applied in order to study the possible contradiction appearing in a fuzzy system (see [3], [21]). In addition, as any other knowledge based system, fuzzy systems may be affected by a lack of information ([15]). In particular, in the case of type-2 fuzzy systems, any fuzzy set **a** ($\mathbf{a}(x) = a$ for all $x \in [0, 1]$) with $a \in [0, 1]$ represents a lack of information. Consequently, it is interesting to consider how these sets (representing lack of information) are transformed through a negation.

In summary, the study of strong negations is essential for both fuzzy sets and their extensions, and particularly for type-2 fuzzy sets. Trillas in [22] studied and characterized strong negations in $[0, 1]$. Bustince et al. in [2] introduce intuitionistic generators in order to build negations in A-IFSs; and Deschrijver et al. in [6] characterized strong intuitionistic negations based on strong negations in $[0, 1]$. Further characteristics for strong negations in A-IFSs and interval-valued fuzzy sets were provided in [2].

The analysis of negations for T2FSs takes place in the framework defined by $\mathbf{M} = [0, 1]^{[0,1]}$, the set of all functions from $[0, 1]$ to $[0, 1]$. These functions constitute the membership functions characterizing type-2 fuzzy sets. In some cases, it is interesting to restrict the analysis to **L**, the set of normal and convex functions in **M**. In [11] a deep study about the negations on T2FSs is presented, including the axioms for negations and strong negations on a bounded partially ordered set (bounded poset). On the basis of Zadeh's Extension Principle, families of negations and strong negations on **L** are defined. Negations on **M** are presented for the first time in [20], where some new negations and strong negations on **L** are also introduced. A family of strong negations on **L** that leave fixed the constant function **1** is presented. This family was built through strong negations and order automorphisms in $[0, 1]$.

As we will concentrate on **L**, the lack of information represented by the set $\mathbf{a}(x) = a$ previously mentioned, now is limited to the case of **1** ($\mathbf{1}(x) = 1$ for all $x \in [0, 1]$). The present paper is focused on the analysis of strong negations in T2FSs, with particular interest in considering how the fuzzy set **1**, that represents a lack of information, is transformed by those strong negations. In a first step, we will explore strong negations in **L** to obtain any possible image of the function **1**

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through them. As the particular case of strong negations mapping **1** to **1** has been previously studied, in this work we will generalize the previous results by considering the main properties of those strong negations in **L** where the image of the function **1** is not this same function.

The paper is organized as follows. In Section 2 we review some definitions, operations and properties on T2FS and on the negations in the set of functions from $[0,1]$ to $[0,1]$, that will be applied in the paper. Section 3 is devoted to obtain the images of the constant function **1** through a strong negation in **L**. Sections 4 and 5 focus on analysing the images obtained in previous section. Finally, Section 6 presents some conclusions.

2 PRELIMINARIES

Throughout the paper, X will denote a non-empty set which will represent the universe of discourse. Additionally, $\leq$ will denote the usual order relation in the lattice of real numbers, and $\wedge$ and $\vee$ the minimum and the maximum operators, respectively.

2.1 Definitions and Properties of T1FSs and T2FSs

Definition 1 *([26]) A type-1 fuzzy set (T1FS) A in a universe X, is characterized by a membership function μ_A,*

$$\mu_A : X \to [0,1],$$

where $\mu_A(x)$ is the membership degree of an element $x \in X$ in the set A.

Definition 2 *([17],[18]) A type-2 fuzzy set (T2FS) A in a universe X, is characterized by a membership function μ_A,*

$$\mu_A : X \to \boldsymbol{M} = [0,1]^{[0,1]} = Map([0,1],[0,1]),$$

that is, $\mu_A(x)$ is a type-1 fuzzy set in the interval $[0,1]$ and also the membership degree of the element $x \in X$ in the set A. Therefore,

$$\mu_A(x) = f_x, \text{ where } f_x : [0,1] \to [0,1].$$

Figure 1 shows an example of a type-2 fuzzy set on the finite universe of discourse $T = \{0,1,2,3,4\}$.

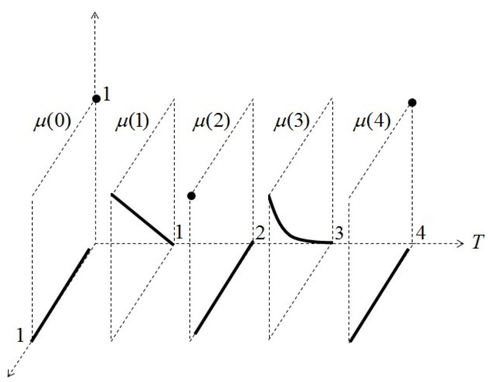

Figure 1. Example of a T2FS

Walker and Walker justify in [23] that the operations on $Map(X, \mathbf{M})$ can be defined naturally from the operations on **M** and have the same properties. Thus, in the same way as in the case of the FSs, where the definitions and properties are given on $([0,1], \leq)$, in this paper we will work on **M**, as all the results are easily and directly extensible to $Map(X, \mathbf{M})$.

Moreover, we restrict to type-2 fuzzy sets in the special case in which the membership degrees are in **L**, the set of normal and convex functions of **M**. There are several reasons to do so. First, membership degrees represent linguistic labels of the TRUTH variable, so it is not uncommon to require them to be convex and normal. Furthermore, it has been pointed out that this set **L** will contain a bounded and complete lattice structure, and as consequence t-norms, t-conorms, aggregation operators and specifically negations can be properly constructed (see [11], [19]).

Definition 3 *([13]) A function $f \in \boldsymbol{M}$ is normal if $sup\{f(x) : x \in [0,1]\} = 1$.*

It is important to notice that with this definition it is possible to find a normal function in **M** such that $f(x) < 1 \; \forall x \in [0,1]$.

Definition 4 *A function $f \in \boldsymbol{M}$ is convex if for any $x \leq y \leq z$, it holds that $f(y) \geq f(x) \wedge f(z)$.*

The following definition and theorem are needed to analyse the operations in the set **M**.

Definition 5 *([8], [9], [23]) If $f \in \boldsymbol{M}$, we define $f^L, f^R \in \boldsymbol{M}$ as*

$$f^L(x) = sup\{f(y) \; : y \leq x\}, \;\; f^R(x) = sup\{f(y) \; : y \geq x\}$$

Note that f^L and f^R are increasing and decreasing, respectively (see Figure 2), $f \leq f^L, f \leq f^R$, for all $f \in \mathbf{M}$ ([23]), where $\leq$ is the usual order in the set of functions ($f \leq g$ if and only if $f(x) \leq g(x)$, for all x).

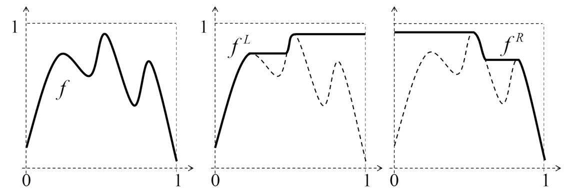

Figure 2. Example of f^L and f^R

In previous papers two partial orders on **M** ($\sqsubseteq$ and $\preceq$) were defined. In general, these partial orders are not equivalent ([17], [23]), and **M** does not have a lattice structure with either of the two orders. However, in **L** these partial orders are equivalent, that is $\sqsubseteq \equiv \preceq$; and $(\mathbf{L}, \sqsubseteq)$ is a bounded and complete lattice (see [9], [10], [18], [23]). The following characterization of the partial order $\sqsubseteq$ on **L** will be useful for establishing the results of this work.

Theorem 1 *([9], [10]) Let $f, g \in \boldsymbol{L}$.*

$$f \sqsubseteq g \text{ if and only if } g^L \leq f^L \text{ and } f^R \leq g^R$$

Let us remind now some properties of the functions in the lattice $(\mathbf{L}, \sqsubseteq)$. We denote $L_c = \{f \in \mathbf{L} : f \; increasing\}$, $L_d = \{f \in \mathbf{L} : f \; decreasing\}$, and $\mathbf{1} \in \mathbf{L}$ the constant function such that $\mathbf{1}(x) = 1$ for all $x \in [0,1]$.

Proposition 1 *([23]) Let $f, g \in \boldsymbol{L}$.*

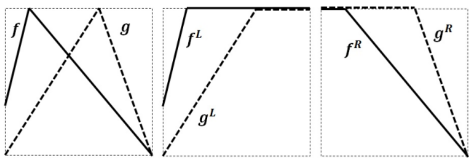

Figure 3. Example where $f \sqsubseteq g$

1. $(f \in L_c) \Leftrightarrow f = f^L \Leftrightarrow f(1) = 1$, *and* $(f \in L_d) \Leftrightarrow f = f^R \Leftrightarrow f(0) = 1$.
2. *If* $f, g \in L_c$, *it holds that* $f \sqsubseteq g \Leftrightarrow g \leq f$.
3. *If* $f, g \in L_d$, *it holds that* $f \sqsubseteq g \Leftrightarrow f \leq g$.
4. $f = f^L \wedge f^R$, *and* $f^L \vee f^R = \mathbf{1}$

Note that $\wedge$ and $\vee$ operations have the usual meaning in the set of functions, that is, $(f \wedge g)(x) = f(x) \wedge g(x)$, and $(f \vee g)(x) = f(x) \vee g(x)$.

The following lemma describes some properties of the functions in the lattice $(\mathbf{L}, \sqsubseteq)$.

Lemma 1 *([5]) Let* $f, g \in \mathbf{L}$

1. $f \sqsubseteq g$ *if and only if* $f^L \sqsubseteq g^L$ *and* $f^R \sqsubseteq g^R$.
2. *If* $f \in \mathbf{L}$, *then* $f^R \sqsubseteq f \sqsubseteq f^L$.
3. $f \in L_d \Leftrightarrow f \sqsubseteq \mathbf{1}$.
4. $f \in L_c \Leftrightarrow \mathbf{1} \sqsubseteq f$.
5. $f^L = f^R \Leftrightarrow f = \mathbf{1}$.

Definition 6 *([23]) Let* $[a, b] \subseteq [0, 1]$. *The characteristic function of* $[a, b]$ *is* $\overline{[a, b]} : [0, 1] \to [0, 1]$, *where*

$$\overline{[a,b]}(x) = \begin{cases} 1, & \text{if } x \in [a,b], \\ 0, & \text{if } x \notin [a,b], \end{cases}$$

and the characteristic function of a number $a \in [0, 1]$ *(a singleton) is* $\bar{a} = \overline{[a, a]}$.

Note that $\overline{0}$ and $\overline{1}$ are the minimum and maximum, respectively, of the bounded lattice $(\mathbf{L}, \sqsubseteq)$.

2.2 Negations on L

This subsection is devoted to review the studies carried out in [11], [20] and [5] on the negations in the framework of the T2FSs with membership degrees in $\mathbf{L}$.

Firstly, let us recall the definition of negation in $([0, 1], \leq)$.

Definition 7 *A function* $n : [0, 1] \to [0, 1]$ *is a negation if it is decreasing respect to the order* $\leq$ *and satisfies* $n(0) = 1$ *and* $n(1) = 0$. *If, in addition,* $n(n(x)) = x$ *for all* $x \in [0, 1]$, *then it is said to be a strong negation.*

Definition 7 suggests us an extension to any partially ordered set (poset) with minimum and maximum elements (bounded). In this sense, Hernández et al. ([11]) introduced negations in this algebraic structure and gave some negations in $\mathbf{L}$.

Definition 8 *([11]) Let* A *be a set and* $\leq_A$ *be a partial order in* A *such that* $(A, \leq_A)$ *has a minimum element* $0_{\leq_A}$ *and a maximum element* $1_{\leq_A}$. *A negation in* $(A, \leq_A)$ *is a decreasing function* $N : A \to A$ *such that* $N(0_{\leq_A}) = 1_{\leq_A}$, *and* $N(1_{\leq_A}) = 0_{\leq_A}$. *If, additionally,* $N(N(x)) = x$ *holds for all* $x \in A$, *it is said to be a strong negation.*

Proposition 2 *([1]) Let* $(A, \leq_A)$ *be a lattice. Given a strong negation* $N : A \to A$, *for any* $a, b \in A$ *it holds that* $N(\inf\{a, b\}) = sup\{N(a), N(b)\}$ *and* $N(sup\{a, b\}) = \inf\{N(a), N(b)\}$.

In [20] we have obtained a family of negations on $(\mathbf{L}, \sqsubseteq)$ that transform singletons into singletons and are closed on the set of characteristic functions of closed intervals.

Theorem 2 *([20]) Let* n *be a strong negation in* $([0, 1], \leq)$, *let* α *be an order automorphism in* $([0, 1], \leq)$, *and let* $N_{n,\alpha} : \mathbf{L} \to \mathbf{L}$ *be the operation defined as* $N_{n,\alpha}(f) = (\alpha \circ f^R \circ n) \wedge (\alpha^{-1} \circ f^L \circ n)$. *Then*

a) $N_{n,\alpha}$ *is a strong negation in* $\mathbf{L}$.
b) $N_{n,\alpha}(\overline{a}) = \overline{n(a)}$ *and* $N_{n,\alpha}(\overline{[a, b]}) = \overline{[n(b), n(a)]}$.
c) $N_{n,\alpha}(\overline{p}) = \overline{p}$ *and* $N_{n,\alpha}(\mathbf{1}) = \mathbf{1}$, *being* p *the fixed point of the strong negation in* $[0, 1]$, n, *and* $\mathbf{1}$ *the constant function with value 1 at any* $x \in [0, 1]$.

Moreover, in [5] the authors obtained a characterization of the strong negations on $\mathbf{L}$ that leave the constant function $\mathbf{1}$ fixed.

Theorem 3 *([5])* $N : L \to L$ *is a strong negation in* $(\mathbf{L}, \sqsubseteq)$ *such that* $N(\mathbf{1}) = \mathbf{1}$ *if and only if there exists a strong negation* n *and an order automorphism* α *in* $([0, 1], \leq)$, *such that* $N(f) = (\alpha \circ f^R \circ n) \wedge (\alpha^{-1} \circ f^L \circ n)$.

3 NEGATION OF CONSTANT FUNCTION 1 BY STRONG NEGATIONS ON L

This section will analyze the negation of the constant function $\mathbf{1}$ by a strong negation, to show that the possible results are restricted to:

- the function $\mathbf{1}$,
- a singleton different from $\overline{0}$ and $\overline{1}$,
- a function taking a constant value a in $(0, 1]$ $(a \neq 1)$ and the value 1 at 0, or
- a function taking a constant value a in $[0, 1)$ $(a \neq 1)$ and the value 1 at 1.

To do so we will consider the following lemma, jointly with Propositions 1 and 2, and Lemma 1.

Lemma 2 *Let* $f, g, h \in L$. *Then*

1. *If* $f \sqsubseteq h$, $g \sqsubseteq h$, $f^L = h^L$ *and* $g^R = h^R$, *then* $sup_{\sqsubseteq}\{f, g\} = h$.
2. *If* $h \sqsubseteq f$, $h \sqsubseteq g$, $h^L = f^L$ *and* $h^R = g^R$, *then* $\inf_{\sqsubseteq}\{f, g\} = h$.

Proof Let $f \sqsubseteq h$ and $g \sqsubseteq h$, then $f \sqsubseteq sup_{\sqsubseteq}\{f, g\} \sqsubseteq h$, and $f^L \sqsubseteq sup_{\sqsubseteq}\{f, g\}^L \sqsubseteq h^L$. So, $h^L = sup_{\sqsubseteq}\{f, g\}^L$. Moreover, $g \sqsubseteq sup_{\sqsubseteq}\{f, g\} \sqsubseteq h$, and $g^R \sqsubseteq sup_{\sqsubseteq}\{f, g\}^R \sqsubseteq h^R$. So, $h^R = sup_{\sqsubseteq}\{f, g\}^R$.

Therefore,

$$h = h^L \wedge h^R = sup_{\sqsubseteq}\{f, g\}^L \wedge sup_{\sqsubseteq}\{f, g\}^R = sup_{\sqsubseteq}\{f, g\}$$

The proof of the second property is analogous. ■

Definition 9 *For any* $m \in (0, 1)$ *and* $s \in (0, 1)$, *we define the functions* $f_{m,1} \in L$ *and* $f_{s,0} \in L$ *(Figure 4) as follows:*

$$f_{m,1}(x) = \begin{cases} m, & \text{if } x \in [0,1), \\ 1, & \text{if } x = 1. \end{cases}$$

and

$$f_{s,0}(x) = \begin{cases} 1, & \text{if } x = 0, \\ s, & \text{if } x \in (0,1]. \end{cases}$$

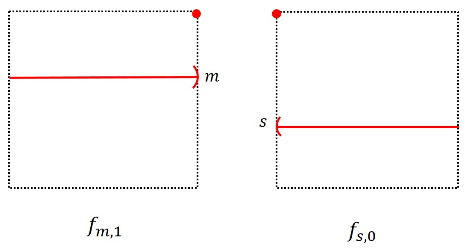

Figure 4. Functions $f_{m,1}$ and $f_{s,0}$

Theorem 4 *If $N : \boldsymbol{L} \to \boldsymbol{L}$ is a strong negation in $(\mathbf{L}, \sqsubseteq)$, then $N(\boldsymbol{1}) = \boldsymbol{1}$ or $N(\boldsymbol{1}) = \overline{a}$ with $a \in (0,1)$, or $N(\boldsymbol{1}) = f_{m,1}$ for some $m \in (0,1)$ or $N(\boldsymbol{1}) = f_{s,0}$ for some $s \in (0,1)$.*

Proof First, being N a strong negation, it is obvious that $N(\mathbf{1}) \neq \overline{0}$ and $N(\mathbf{1}) \neq \overline{1}$.

Now we will proceed by reduction to absurdity.

Let us suppose that $N(\mathbf{1}) \neq \mathbf{1}$, $N(\mathbf{1}) \neq \overline{a}$ with $a \in (0,1)$, $N(\mathbf{1}) \neq f_{m,1}$ for all $m \in (0,1)$ and $N(\mathbf{1}) \neq f_{s,0}$ for all $s \in (0,1)$.

1. If $N(\mathbf{1}) \in L_d$ and $N(\mathbf{1})$ is continuous in $x = 0$, then $N(\mathbf{1}) \sqsubseteq \mathbf{1}$, $N(\mathbf{1}) \neq \mathbf{1}$, $(N(\mathbf{1}))(0) = 1$ and $(N(\mathbf{1}))(1) < 1$, and $sup\{(N(\mathbf{1}))(x) : x \in (0,1]\} = 1$, since $N(\mathbf{1})$ continuous in $x = 0$ and $(N(\mathbf{1}))(0) = 1$.

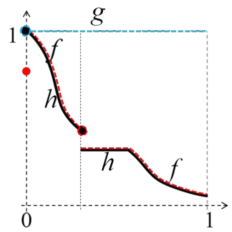

Figure 5. Function $h = N(\mathbf{1}) \neq \mathbf{1}$ decreasing, continuous in $x = 0$ and not constant in $(0,1]$.

Let us take the functions $g = \mathbf{1}$ and $f : [0,1] \to [0,1]$ such that $f(0) < 1$ and $f(x) = (N(\mathbf{1}))(x)$ for all $x \in (0,1]$. Let us observe that $f \in \mathbf{L}$ because $sup\{f(x) : x \in (0,1]\} = sup\{(N(1))(x) : x \in (0,1]\} = 1$. It is verified that $N(\mathbf{1}) \sqsubsetneq \mathbf{1} = g$ (so $N(g) \in L_d$ and $N(g) \neq \mathbf{1}$), $N(\mathbf{1}) \sqsubsetneq f$ (so $N(f) \in L_d$ and $N(f) \neq \mathbf{1}$), $g^L = \mathbf{1} = (N(\mathbf{1}))^L$ and $f^R = \mathbf{1} = (N(\mathbf{1}))^R = N(\mathbf{1})$. Therefore, by Lemma 2, $\inf_{\sqsubseteq}\{g, f\} = N(\mathbf{1})$, and so $\mathbf{1} = N(N(\mathbf{1})) = N(\inf_{\sqsubseteq}\{g, f\}) = sup_{\sqsubseteq}\{N(g), N(f)\} = N(g) \vee N(f) \neq \mathbf{1}$, since $N(g)$ and $N(f)$ are decreasing and $N(g) \neq \mathbf{1}$, $N(f) \neq \mathbf{1}$, and so $(N(g))(1) < 1$ and $(N(f))(1) < 1$. (see Figure 5 where $h \sqsubsetneq g$, $h \sqsubsetneq f$ and $\inf_{\sqsubseteq}\{g, f\} = h$)

Let us observe the function f defined above belongs to $\mathbf{L}$, as $N(1)$ is decreasing and continuous at $x = 0$; in other case the proof does not work. For example, if $N(1) = \frac{1}{2}, \forall x \in (0,1]$ and $(N(1))(0) = 1$, then $f \notin \mathbf{L}$, as $sup f = f(0) < 1$.

2. If $N(\mathbf{1}) \in L_d$, $N(\mathbf{1})$ is not continuous in $x = 0$ and $N(\mathbf{1})$ is not constant in $(0,1]$, then $N(\mathbf{1}) \sqsubsetneq \mathbf{1}$, $(N(\mathbf{1}))(0) = 1$ and $(N(\mathbf{1}))(1) < 1$, and there are $x_1, x_2 \in (0,1]$ such that $x_1 < x_2$ and $(N(\mathbf{1}))(x_2) < (N(\mathbf{1}))(x_1) < 1$.

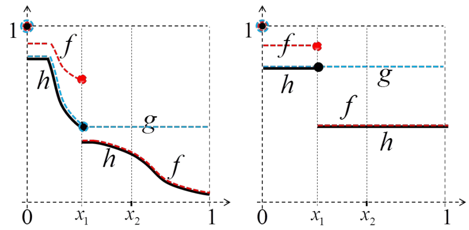

Figure 6. Function $h = N(\mathbf{1})$ decreasing, not continuous in $x = 0$ and not constant in $(0,1]$.

Let us take the functions $f, g : [0,1] \to [0,1]$ such that

$$f(x) = \begin{cases} \frac{1}{2}(1 + (N(\mathbf{1}))(x)), & \text{if } x \in [0, x_1], \\ (N(\mathbf{1}))(x), & \text{if } x \in (x_1, 1], \end{cases}$$

and

$$g(x) = \begin{cases} (N(\mathbf{1}))(x), & \text{if } x \in [0, x_1], \\ (N(\mathbf{1}))(x_1), & \text{if } x \in [x_1, 1], \end{cases}$$

It is verified that $f \in L_d$ and $N(\mathbf{1}) \sqsubsetneq f \sqsubsetneq \mathbf{1}$, $N(\mathbf{1}) \sqsubsetneq N(f) \sqsubsetneq \mathbf{1}$, $N(f) \in L_d$ and $(N(f))(1) < 1$.
And $g \in L_d$ and $N(\mathbf{1}) \sqsubsetneq g \sqsubsetneq \mathbf{1}$, $N(\mathbf{1}) \sqsubsetneq N(g) \sqsubsetneq \mathbf{1}$, $N(g) \in L_d$ and $(N(g))(1) < 1$.
Moreover, $\inf_{\sqsubseteq}\{f, g\} = f \wedge g = N(\mathbf{1})$, and so $\mathbf{1} = N(N(\mathbf{1})) = N(\inf_{\sqsubseteq}\{f, g\}) = sup_{\sqsubseteq}\{N(f), N(g)\} = N(f) \vee N(g) \neq \mathbf{1}$, since $(N(f))(1) < 1$ and $(N(g))(1) < 1$. (See Figure 6 where $h \sqsubsetneq g$, $h \sqsubsetneq f$ and $\inf_{\sqsubseteq}\{g, f\} = g \wedge f = h$).

3. If $N(\mathbf{1}) \in L_c$ and $N(\mathbf{1})$ continuous in $x = 1$, then $\mathbf{1} \sqsubsetneq N(\mathbf{1})$, $(N(\mathbf{1}))(0) < 1$ and $(N(\mathbf{1}))(1) = 1$ (since $N(\mathbf{1}) \neq \mathbf{1}$), and $sup\{(N(\mathbf{1}))(x) : x \in [0,1)\} = 1$ (since $N(\mathbf{1})$ continuous in $x = 1$ and $(N(\mathbf{1}))(1) = 1$).
Let g and f be the functions such that $g = \mathbf{1}$ and $f(x) = (N(\mathbf{1}))(x)$ for all $x \in [0,1)$ and $f(1) < 1$. Again $f \in \mathbf{L}$ because $sup\{f(x) : x \in [0,1)\} = sup\{(N(\mathbf{1}))(x) : x \in [0,1)\} = 1$.
It is verified that $g = \mathbf{1} \sqsubsetneq N(\mathbf{1})$, $N(g) \in L_c$ and $N(g) \neq \mathbf{1}$, $f \sqsubsetneq N(\mathbf{1})$ (so $N(f) \in L_c$ and $N(f) \neq \mathbf{1}$), $g^R = \mathbf{1} = (N(\mathbf{1}))^R$ and $f^L = (N(\mathbf{1}))^L = N(\mathbf{1})$.
Therefore, $sup_{\sqsubseteq}\{f, g\} = N(\mathbf{1})$, and so $\mathbf{1} = N(N(\mathbf{1})) = N(sup_{\sqsubseteq}\{g, f\} = \inf_{\sqsubseteq}\{N(g), N(f)\} = N(g) \vee N(f) \neq \mathbf{1}$, since $N(g)$ and $N(f)$ are increasing and $N(g) \neq \mathbf{1}$, $N(f) \neq \mathbf{1}$, and so $(N(g))(0) < 1$ and $(N(g))(0) < 1$.
4. In a similar way as in the decreasing case, we can prove that $N(\mathbf{1})$ increasing, non-continuous in $x = 1$ and $N(\mathbf{1})$ not constant in $[0,1)$ is not possible.
5. If $N(\mathbf{1}) \notin L_d \cup L_c$ then $(N(\mathbf{1}))(0) < 1$ and $(N(\mathbf{1}))(1) < 1$, according to Proposition 1.
Let us first remember that a function f is normal if and only if there exists $x_1 \in [0,1]$ satisfying at least one of the following three conditions:

   i) $f(x_1) = 1$,

   ii) $lim_{x \to x_1^+} f(x) = 1$,

   iii) $lim_{x \to x_1^-} f(x) = 1$.

   Considering now this three cases we have:

   i) $x_1 \in (0,1)$ such that $(N(\mathbf{1}))(x_1) = 1$ ($x_1 = 0$ or $x_1 = 1$ is not possible as $N(\mathbf{1}) \notin L_d \cup L_c$). We have several cases:

   a) There is $x \in (x_1, 1]$ and $(N(\mathbf{1}))(x) > 0$. Let us take the functions $f, g : [0,1] \to [0,1]$ such that

$$f(x) = \begin{cases} (N(\mathbf{1}))(x), & \text{if } x \in [0, x_1], \\ \frac{1}{2}(N(\mathbf{1}))(x), & \text{if } x \in (x_1, 1], \end{cases}$$

and

$$g(x) = \begin{cases} \frac{1}{2}(1 + (N(\mathbf{1}))(x)), & \text{if } x \in [0, x_1], \\ (N(\mathbf{1}))(x), & \text{if } x \in [x_1, 1], \end{cases}$$

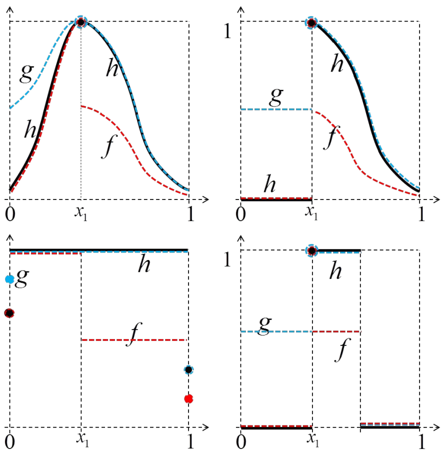

Figure 7. Function $h = N(\mathbf{1}) \notin L_c \cup L_d$, $h(x_1) = 1$ and $h(x) > 0$ for some $x \in (x_1, 1]$.

It is verified that, $f \subsetneq N(\mathbf{1})$, $g \subsetneq N(\mathbf{1})$, $f^L = N(\mathbf{1})^L$ and $g^R = N(\mathbf{1})^R$. Then $sup_{\sqsubseteq}\{f, g\} = N(\mathbf{1})$ by Lemma 2, it is $\mathbf{1} = N(N(\mathbf{1})) = N(sup_{\sqsubseteq}\{f, g\}) = \inf_{\sqsubseteq}\{N(f), N(g)\}$ (See Figure 7 where $g \subsetneq h$, $f \subsetneq h$, $g^R = h^R$ and $f^L = h^L$). However, as $f \subsetneq N(\mathbf{1})$ and $g \subsetneq N(\mathbf{1})$, we have $\mathbf{1} = N(N(\mathbf{1})) \subsetneq N(f)$ and $\mathbf{1} \subsetneq N(g)$. And so, $N(f), N(g) \in L_c$, $N(f), N(g) \neq \mathbf{1}$, $(N(f))(0) < 1$ and $(N(g))(0) < 1$. Then, $\inf_{\sqsubseteq}\{N(f), N(g)\} = N(f) \vee N(g) \neq \mathbf{1}$.

b) There is $x \in [0, x_1)$ and $(N(\mathbf{1}))(x) > 0$. Let us take the functions $f, g : [0,1] \to [0,1]$ such that

$$f(x) = \begin{cases} (N(\mathbf{1}))(x), & \text{if } x \in [0, x_1], \\ \frac{1}{2}(1 + N(\mathbf{1}))(x), & \text{if } x \in (x_1, 1], \end{cases}$$

and

$$g(x) = \begin{cases} \frac{1}{2}(N(\mathbf{1}))(x), & \text{if } x \in [0, x_1), \\ (N(\mathbf{1}))(x), & \text{if } x \in [x_1, 1], \end{cases}$$

(See Figure 8 where $h \subsetneq g$, $h \subsetneq f$, $g^R = h^R$ and $f^L = h^L$).

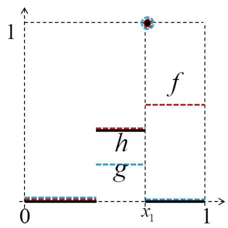

Figure 8. Function $h = N(\mathbf{1}) \notin L_c \cup L_d$, $h(x_1) = 1$ and $h(x) > 0$ for some $x \in [0, x_1)$

It is verified that, $N(\mathbf{1}) \subsetneq f$, $N(\mathbf{1}) \subsetneq g$, $f^L = N(\mathbf{1})^L$ and $g^R = N(\mathbf{1})^R$. Then $\inf_{\sqsubseteq}\{f, g\} = N(\mathbf{1})$ by Lemma 2, and $\mathbf{1} = N(N(\mathbf{1})) = N(\inf_{\sqsubseteq}\{f, g\}) = sup_{\sqsubseteq}\{N(f), N(g)\}$. However, as $N(\mathbf{1}) \subsetneq f$ and $N(\mathbf{1}) \subsetneq g$, we have $N(f) \subsetneq N(N(\mathbf{1})) = \mathbf{1}$ and $N(g) \subsetneq \mathbf{1}$. And so, $N(f), N(g) \in L_d$, $N(f), N(g) \neq \mathbf{1}$, $(N(f))(1) < 1$ and $(N(g))(1) < 1$. Then, $sup_{\sqsubseteq}\{N(f), N(g)\} = N(f) \vee N(g) \neq \mathbf{1}$.

ii) Let $x_1 \in [0, 1)$ such that $lim_{x \to x_1^+} f(x) = 1$, and $(N(\mathbf{1}))(x_1) < 1$. Therefore, there is $x_2 \in (x_1, 1]$ such that $(N(\mathbf{1}))(x) > 0$ for all $x \in (x_1, x_2]$.
Let us take the functions $f, g : [0,1] \to [0,1]$ such that

$$f(x) = \begin{cases} (N(\mathbf{1}))(x), & \text{if } x \in [0, x_2), \\ \frac{1}{2}(N(\mathbf{1}))(x), & \text{if } x \in [x_2, 1]. \end{cases}$$

and

$$g(x) = \begin{cases} \frac{1}{2}(N(\mathbf{1}))(x), & \text{if } x \in [0, x_1), \\ (N(\mathbf{1}))(x), & \text{if } x \in (x_1, 1], \end{cases}$$

(see Figure 9 where $g \subsetneq h$, $f \subsetneq h$, $g^R = h^R$ and $f^L = h^L$).

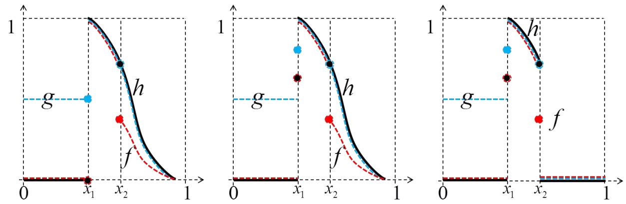

Figure 9. Function $h = N(\mathbf{1}) \notin L_c \cup L_d$, $lim_{x \to x_1^+} h(x) = 1$ and $h(x_2) > 0$ with $x_2 \in (x_1, 1]$

$f \subsetneq N(\mathbf{1})$, $g \subsetneq N(\mathbf{1})$, $f^L = N(\mathbf{1})^L$ and $g^R = N(\mathbf{1})^R$. Then $sup_{\sqsubseteq}\{f, g\} = N(\mathbf{1})$ and so $\mathbf{1} = N(sup_{\sqsubseteq}\{f, g\}) = \inf_{\sqsubseteq}\{N(f), N(g)\} = N(f) \vee N(g)$, since $N(f), N(g) \in L_c$, but $N(f) \vee N(g) \neq \mathbf{1}$, as $(N(f))(0) < 1$ and $(N(g))(0) < 1$, since $N(f)$ and $N(g)$ are increasing and different from $\mathbf{1}$.

iii) Let $x_1 \in (0, 1]$ such that $lim_{x \to x_1^-} f(x) = 1$, and $(N(\mathbf{1}))(x_1) < 1$. Therefore, there is $x_2 \in [0, x_1)$ such that $(N(\mathbf{1}))(x) > 0$ for all $x \in [x_2, x_1)$.
Let us take the functions $f, g : [0,1] \to [0,1]$ such that

$$f(x) = \begin{cases} (N(\mathbf{1}))(x), & \text{if } x \in [0, x_1), \\ \frac{1}{2}(1 + N(\mathbf{1})(x)), & \text{if } x \in [x_1, 1]. \end{cases}$$

and

$$g(x) = \begin{cases} \frac{1}{2}(N(\mathbf{1}))(x), & \text{if } x \in [0, x_1), \\ (N(\mathbf{1}))(x), & \text{if } x \in [x_1, 1], \end{cases}$$

(see Figure 10 where $h \subsetneq g$, $h \subsetneq f$, $g^R = h^R$ and $f^L = h^L$). $N(\mathbf{1}) \subsetneq f$, $N(\mathbf{1}) \subsetneq g$, $f^L = N(\mathbf{1})^L$ and $g^R = N(\mathbf{1})^R$. Then $\inf_{\sqsubseteq}\{f, g\} = N(\mathbf{1})$ and so $\mathbf{1} = N(\inf_{\sqsubseteq}\{f, g\}) = sup_{\sqsubseteq}\{N(f), N(g)\} = N(f) \vee N(g)$, since $N(f), N(g) \in L_d$, but $N(f) \vee N(g) \neq \mathbf{1}$, as $(N(f))(1) < 1$ and $(N(g))(1) < 1$, since $N(f)$ and $N(g)$ are decreasing and different from $\mathbf{1}$.

Therefore, in any case different from those indicated in the Theorem, a contradiction is reached. ■

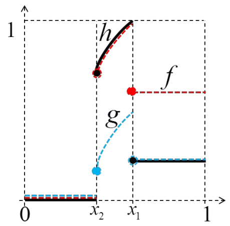

Figure 10. Function $h = N(\mathbf{1}) \notin L_c \cup L_d$, $lim_{x \to x_1^-} h(x) = 1$ and $h(x_2) > 0$ with $x_2 \in [0, x_1)$

4 STRONG NEGATIONS WITH $N(\mathbf{1}) = f_{m,1}$ OR $N(\mathbf{1}) = f_{s,0}$

Now, we will present some properties of strong negations such that $N(\mathbf{1}) = f_{m,1}$ for some $m \in (0, 1)$. In particular, we will find the images of the singletons, and the functions $f_{k,1}$ and $f_{k,0}$.

4.1 Images of the singletons

Lemma 3 *If $a \in (0, 1)$, there are no $f, g \in \boldsymbol{L}$, with $f \neq \overline{a} \neq g$, such that $sup_{\sqsubseteq}\{f, g\} = \overline{a}$ or $\inf_{\sqsubseteq}\{f, g\} = \overline{a}$.*

Proof Let us suppose that $sup_{\sqsubseteq}\{f, g\} = \overline{a}$ with $f \neq \overline{a} \neq g$ (in case $\inf_{\sqsubseteq}\{f, g\} = \overline{a}$ the proof is similar).

As $f \sqsubseteq \overline{a}$ and $g \sqsubseteq \overline{a}$, $f^R \leq \overline{a}^R$, $g^R \leq \overline{a}^R$ and we have $f(x) = g(x) = 0$ for all $x > a$.

Furthermore, $(sup_{\sqsubseteq}\{f, g\})(a) = (\overline{a})(a) = 1$, and then $f(a) = 1$ or $g(a) = 1$.

i) If $f(a) = g(a) = 1$, there exists $m \in [0, a)$ such that $f(m) > 0$, so $f(x) > 0$ $\forall x \in [m, a)$. It should be $g(x) = 0$ $\forall x \in [m, a)$, and, taking into account that $g \in \mathbf{L}$, $g(x) = 0$ $\forall x \in [0, a)$. Finally, $g = \overline{a}$, attaining a contradiction.

ii) If, for example, $f(a) < 1$, there exists $m \in [0, a)$ such that $f(m) > f(a)$, and then $f(x) > f(a)$ $\forall x \in [m, a)$. So, $g(x) = 0$ $\forall x \in [m, a)$, $g(x) = 0$ $\forall x \in [0, a)$, and $g = \overline{a}$, which is also a contradiction. ∎

Remark 1 *Consequently, there are no $f, g \in \boldsymbol{L}$, with $f \neq \overline{a} \neq g$, such that $N(sup_{\sqsubseteq}\{f, g\}) = N(\overline{a})$, or, such that $\inf_{\sqsubseteq}\{N(f), N(g)\} = N(\overline{a})$. In other words, there are no $h, l \in \boldsymbol{L}$ with $h \neq \overline{a} \neq l$, satisfying $\inf_{\sqsubseteq}\{h, l\} = N(\overline{a})$.*

In a similar way, there are no $h, l \in \boldsymbol{L}$ with $h \neq \overline{a} \neq l$, satisfying $sup_{\sqsubseteq}\{h, l\} = N(\overline{a})$.

Theorem 5 *If $N : \boldsymbol{L} \to \boldsymbol{L}$ is a strong negation with $N(\boldsymbol{1}) = f_{m,1}$ for some $m \in (0, 1)$, the image of a singleton is also a singleton.*

Proof Let $a \in (0, 1)$. Using the previous Lemma, and with a proof similar to that of Theorem 4, it could be obtained that
$N(\overline{a}) = \mathbf{1}$, or
$N(\overline{a}) = f_{k,1}$ for some $k \in (0, 1)$, or
$N(\overline{a}) = f_{s,0}$ for some $s \in (0, 1)$, or
$N(\overline{a}) = \overline{b}$ for some $b \in (0, 1)$.

1. If $N(\overline{a}) = \mathbf{1}$, $N(N(\overline{a})) = \overline{a} = N(\mathbf{1}) = f_{m,1}$. Contradiction.
2. Let us suppose that $N(\overline{a}) = f_{k,1}$ for some $k \in (0, 1)$. As $\overline{a}$ is not comparable with $\mathbf{1}$, it should be $N(\overline{a})$ non comparable with $N(\mathbf{1}) = f_{m,1}$. This is a contradiction, as $f_{k,1} \sqsubseteq f_{m,1}$ if $k \geq m$, and $f_{m,1} \sqsubseteq f_{k,1}$ if $k \leq m$.
3. Now, let us suppose that $N(\overline{a}) = f_{s,0}$ for some $s \in (0, 1)$. As $\overline{a}$ is not comparable with $\mathbf{1}$, it should be $N(\overline{a})$ non comparable with $N(\mathbf{1}) = f_{m,1}$. This is a contradiction, as $f_{s,0} \sqsubseteq f_{m,1}$ for all $s \in (0, 1)$.

Then, it should be $N(\overline{a}) = \overline{b}$ for some $b \in (0, 1)$. ∎

Remark 2 *We can note that in this case there exists a unique $b \in (0, 1)$, such that $N(\overline{b}) = \overline{b}$.*

In fact, let us define the function $n : [0, 1] \to [0, 1]$ such that $n(a) = b$ if and only if $N(\overline{a}) = \overline{b}$. Then, it is easy to prove that n is a strong negation in $[0, 1]$, and there is a unique $b \in (0, 1)$ satisfying $n(b) = b$. Then, there is a unique $b \in (0, 1)$ such that $N(\overline{b}) = \overline{b}$. That is, $\overline{b}$ is a 'fixed point' of the negation N.

4.2 Images of the functions $f_{k,1}$ and $f_{k,0}$

Newly, let us begin with a previous Lemma.

Lemma 4 *If $k \in (0, 1)$, there are no $f, g \in \boldsymbol{L}$, with $f \neq f_{k,1} \neq g$, such that $sup_{\sqsubseteq}\{f, g\} = f_{k,1}$ or $\inf_{\sqsubseteq}\{f, g\} = f_{k,1}$.*

Then there are no $f, g \in \mathbf{L}$, with $f \neq f_{k,1} \neq g$, such that $sup_{\sqsubseteq}\{f, g\} = f_{k,1}$, or $N(\inf_{\sqsubseteq}\{N(f), N(g)\}) = N(f_{k,1})$. Equivalently, there are no $h, l \in \mathbf{L}$, with $h \neq N(f_{k,1}) \neq l$ such that $\inf_{\sqsubseteq}\{h, l\} = N(f_{k,1})$.

Theorem 6 *If $N : \boldsymbol{L} \to \boldsymbol{L}$ is a strong negation with $N(\boldsymbol{1}) = f_{m,1}$ for some $m \in (0, 1)$, then for all $k \in (m, 1)$, there exists a $h \in (m, 1)$ such that $N(f_{k,1}) = f_{h,1}$ (see Figure 11, left).*

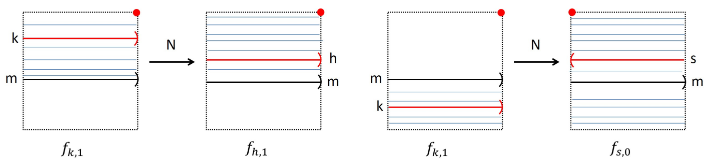

Figure 11. Images of $f_{k,1}$ when $N(\mathbf{1}) = f_{m,1}$

Proof Let $k \in (m, 1)$. Using the previous Lemma, and with a proof similar to that of Theorem 4, it could be obtained that
$N(f_{k,1}) = \mathbf{1}$ or
$N(f_{k,1}) = \overline{a}$ for some $a \in (0, 1)$, or
$N(f_{k,1}) = f_{s,0}$ for some $s \in (0, 1)$, or
$N(f_{k,1}) = f_{h,1}$ for some $h \in (0, 1)$.

1. If $N(f_{k,1}) = \mathbf{1}$, $N(N(f_{k,1})) = f_{k,1} = N(\mathbf{1}) = f_{m,1}$. Contradiction, as $k > m$.
2. Let us suppose that $N(f_{k,1}) = \overline{a}$ for some $a \in (0, 1)$. Then $N(N(f_{k,1})) = f_{k,1} = N(\overline{a}) = \overline{b}$, for some $b \in (0, 1)$. Contradiction, according to Theorem 5.
3. Now, let us suppose that $N(f_{k,1}) = f_{s,0}$ for some $s \in (0, 1)$. As $f_{s,0} \sqsubseteq \mathbf{1}$, $N(\mathbf{1}) = f_{m,1} \sqsubseteq N(f_{s,0}) = f_{k,1}$. But, as $k > m$, it is $f_{k,1} \sqsubsetneq f_{m,1}$.
4. If $N(f_{k,1}) = f_{h,1}$ for some $h \in (0, 1)$ with $h < m$, we have that $f_{m,1} \sqsubsetneq f_{h,1}$, and so $N(f_{h,1}) = f_{k,1} \sqsubsetneq N(f_{m,1}) = \mathbf{1}$, which is false as $\mathbf{1} \sqsubsetneq f_{k,1}$.

Then, only $N(f_{k,1}) = f_{h,1}$, with $h > m$ is possible. ∎

Theorem 7 *If $N : \mathbf{L} \to \mathbf{L}$ is a strong negation with $N(\mathbf{1}) = f_{m,1}$ for some $m \in (0,1)$, then for all $k \in (0,1)$ with $k < m$, there exists a $s \in (0,1)$ such that $N(f_{k,1}) = f_{s,0}$ (see Figure 11, right).*

Proof Let $k \in (0, m)$. In this case, we can also obtain that
$N(f_{k,1}) = \mathbf{1}$ or
$N(f_{k,1}) = \overline{a}$ for some $a \in (0,1)$, or
$N(f_{k,1}) = f_{h,1}$ for some $h \in (0,1)$, or
$N(f_{k,1}) = f_{s,0}$ for some $s \in (0,1)$.

1. If $N(f_{k,1}) = \mathbf{1}$, $N(\mathbf{1}) = f_{k,1} = f_{m,1}$. Contradiction, as $k < m$.
2. If $N(f_{k,1}) = \overline{a}$ for some $a \in (0,1)$, $N(\overline{a}) = \overline{b} = f_{k,1}$, for some $b \in (0,1)$. Contradiction.
3. Let us suppose that $N(f_{k,1}) = f_{h,1}$ for some $h \in (0,1)$. As $f_{m,1} \sqsubseteq f_{k,1}$, we have that $N(f_{k,1}) = f_{h,1} \sqsubseteq N(f_{m,1}) = \mathbf{1}$. But we know that $\mathbf{1} \sqsubseteq f_{h,1}$. Then we attain a contradiction.

Then it must be $N(f_{k,1}) = f_{s,0}$ for some $s \in (0,1)$. ■

In a similar way, it is easy to obtain the following Theorem.

Theorem 8 *If $N : \mathbf{L} \to \mathbf{L}$ is a strong negation with $N(\mathbf{1}) = f_{m,1}$ for some $m \in (0,1)$, then for all $s \in (0,1)$ there exists a $h \in (0,m)$ such that $N(f_{s,0}) = f_{h,1}$ (see Figure 12).*

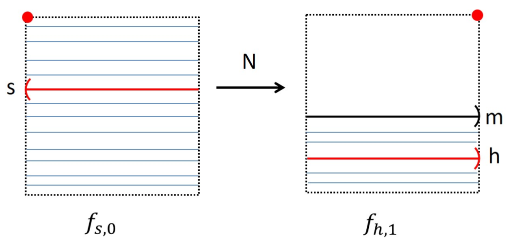

Figure 12. Images of $f_{s,0}$ when $N(\mathbf{1}) = f_{m,1}$

A similar study can be displayed for strong negations that transform the function $\mathbf{1}$ in a function $f_{s,0}$.

Theorem 9 *If $N : \mathbf{L} \to \mathbf{L}$ is a strong negation with $N(\mathbf{1}) = f_{s,0}$ for some $s \in (0,1)$, then (see Figure 13)*

1. *N transforms singletons into singletons. That is, for any $a \in (0,1)$, there exits a $b \in (0,1)$ such that $N(\overline{a}) = \overline{b}$.*
2. *For any $t \in (s,1)$ there exists a $p \in (s,1)$ such that $N(f_{t,0}) = f_{p,0}$.*
3. *For any $t \in (0,s)$ there exists a $m \in (0,1)$ such that $N(f_{t,0}) = f_{m,1}$.*
4. *For any $m \in (0,1)$ there exists a $t \in (0,s)$ such that $N(f_{m,1}) = f_{t,0}$.*

5 STRONG NEGATIONS WITH $N(\mathbf{1}) = \overline{a}$

Finally, we will expose, without proof as it is similar to that of theorems of previous Sections, a result about the images throughout strong negations satisfying $N(\mathbf{1}) = \overline{a}$.

Theorem 10 *If $N : \mathbf{L} \to \mathbf{L}$ is a strong negation with $N(\mathbf{1}) = \overline{a}$ for some $a \in (0,1)$, then (see Figure 14).*

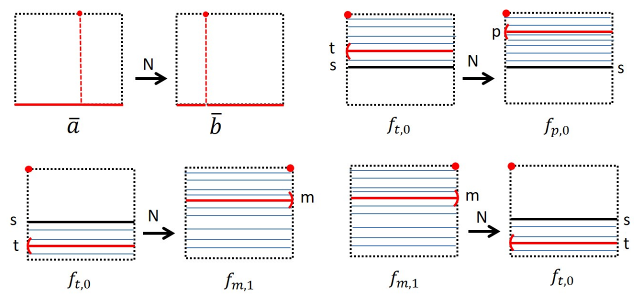

Figure 13. Images when $N(\mathbf{1}) = f_{s,0}$

1. *For any $b \in (0,a)$ there exists a $h \in (0,1)$ such that $N(\overline{b}) = f_{h,1}$.*
2. *For any $b \in (a,1)$ there exists a $s \in (0,1)$ such that $N(\overline{b}) = f_{s,0}$.*
3. *For any $h \in (0,1)$ there exists a $b \in (0,a)$ such that $N(f_{h,1}) = \overline{b}$.*
4. *For any $s \in (0,1)$ there exists a $b \in (a,1)$ such that $N(f_{s,0}) = \overline{b}$.*

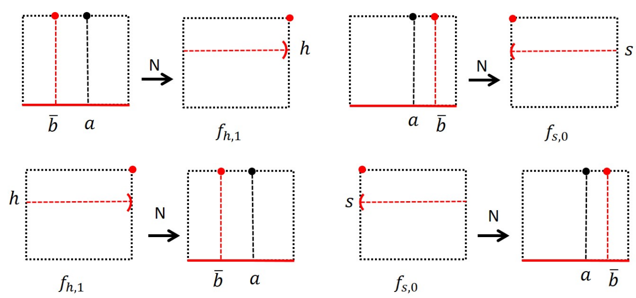

Figure 14. Images when $N(\mathbf{1}) = \overline{a}$

6 CONCLUSION

The present paper focuses on the study of strong negations in the set $\mathbf{L}$, the set of normal and convex functions from $[0,1]$ to $[0,1]$. It considers, in particular, the images of the function $\mathbf{1}$ ($\mathbf{1}(x) = 1$ for all $x \in [0,1]$) through a strong negation. This function has the special interest of representing the lack of information about an event.

In a first step, the possible images of this function have been considered, concluding that there are only four options: the function $\mathbf{1}$ itself, a singleton, a function of the form $f_{m,1}$ (the function taking the value m in $[0,1)$ and the value 1 at $x = 1$), or a function of the form $f_{s,0}$ (the function taking the value 1 at $x = 0$ and the value m in $(0,1]$).

The strong negations leaving fixed the function $\mathbf{1}$ had been characterized, this paper concentrates on the study of the remaining options. In this way, it has been obtained how the images of singletons, functions $f_{m,1}$ and functions $f_{s,0}$ are.

This has meant a breakthrough in the attempt to characterize these strong negations, which will be the purpose of future research.

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REFERENCES

[1] G. Birkhoff, *Lattice Theory*, American Mathematical Society, 1948.
[2] H. Bustince, J. Kacprzyk, and V. Mohedano, 'Intuitionistic fuzzy generators-application to intuitionistic fuzzy complementation', *Fuzzy Sets Syst.*, **114**, 485–504, (2000).
[3] E. E. Castiñeira, C. Torres-Blanc, and S. Cubillo, 'Measuring contradiction on a-ifs defined in finite universes', *Knowledge-Based Syst.*, **24**, 1297–1309, (2011).
[4] O. Castillo, M. A. Sánchez, C. I. González, and G. E. Martínez, 'Review of recent type-2 fuzzy image processing applications', *Information*, **8(3)**, 97–, (2017).
[5] S. Cubillo, C. Torres-Blanc, and P. Hernández-Varela, 'A characterization for some type-2 fuzzy strong negations', *Knowledge-Based Syst.*, (In Press).
[6] G. Deschrijver, C. Cornelis, and E. Kerre, 'Intuitionistic fuzzy connectives revisited', *Proc. of conference IPMU'2002, Annecy (France)*, 1839–1844, (2002).
[7] A. Doostparast Torshizi and M. H. Fazel Zarandi, 'A new cluster validity measure based on general type-2 fuzzy sets: Application in gene expression data clustering', *Knowledge-Based Syst.*, **64**, 81–93, (2014).
[8] Z. Gera and J. Dombi, 'Type-2 implications on non-interactive fuzzy truth values', *Fuzzy Sets Syst.*, **159 (22)**, 3014–3032, (2008).
[9] J. Harding, C. Walker, and E. Walker, 'Lattices of convex normal functions', *Fuzzy Sets Syst.*, **159**, 1061–1071, (2008).
[10] J. Harding, C. Walker, and E. Walker, 'Convex normal functions revisited', *Fuzzy Sets Syst.*, **161**, 1343–1349, (2010).
[11] P. Hernández, S. Cubillo, and C. Torres-Blanc, 'Negations on type-2 fuzzy sets', *Fuzzy Sets Syst.*, **252**, 111–124, (2014).
[12] B. Q. Hu and C. K. Kwong, 'On type-2 fuzzy sets and their t-norm operations', *Information Sciences*, **255**, 58–81, (2014).
[13] G.J. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Prentice Hall, 1995.
[14] O. Linda and M. Manic, 'Monotone centroid flow algorithm for type reduction of general type-2 fuzzy sets', *IEEE Trans. Fuzzy Syst.*, **20(5)**, 805–819, (2012).
[15] N. Madrid and M. Ojeda-Aciago, 'Measuring inconsistency in fuzzy answer set semantics', *IEEE Trans. Fuzzy Systems*, **19**, 605–622, (2011).
[16] J. Mendel and R. Jhon, 'Type-2 fuzzy sets made simple', *IEEE Trans. Fuzzy Syst.*, **10(2)**, 117–127, (1981).
[17] M. Mizumoto and K. Tanaka, 'Some properties of fuzzy sets of type-2', *Inf. Control*, **31**, 312–340, (1976).
[18] M. Mizumoto and K. Tanaka, 'Fuzzy sets of type-2 under algebraic product and algebraic sum', *Fuzzy Sets Syst.*, **5**, 277–290, (1981).
[19] C. Torres-Blanc, S. Cubillo, and P. Hernández, 'Aggregation operators on type-2 fuzzy sets', *Fuzzy Sets Syst.*, **324**, 74–90, (2017).
[20] C. Torres-Blanc, S. Cubillo, and P. Hernández, 'New negations on the membership functions of type-2 fuzzy sets', *IEEE Trans. Fuzzy Syst.*, **27(7)**, 1397–1406, (2019).
[21] C. Torres-Blanc, P. Hernández, and S. Cubillo, 'Self-contradiction for type-2 fuzzy sets whose membership degrees are normal and convex functions', *Fuzzy Sets Syst.*, **352**, 73–91, (2018).
[22] E. Trillas, *On negation functions in fuzzy set theory*, 31–43, Advances of Fuzzy Logic, S. Barro et altri, Santiago de Compostela, 1998. Original version in Spanish (Stochastica, 1979).
[23] C. Walker and E. Walker, 'The algebra of fuzzy truth values', *Fuzzy Sets Syst.*, **149**, 309–347, (2005).
[24] C. Walker and E. Walker, 'Type-2 operations on finite chains', *Fuzzy Sets Syst.*, **236**, 33–49, (2014).
[25] A.J. Yuste, A. Triviño, and E. Casilari, 'Type-2 fuzzy decision support system to optimise manet integration into infrastructure-based wireless systems', *Expert Syst. Appl.*, **40**, 2552–2567, (2013).
[26] L. Zadeh, 'Fuzzy sets', *Information Control*, **20**, 301–312, (1965).
[27] L. Zadeh, 'The concept of a linguistic variable and its application to approximate reasoning-i', *Information Sciences*, **8**, 199–249, (1975).
[28] T. Zhao, J. Xiao, J. Ding, X. Deng, and S. Wang, 'Relaxed stability conditions for interval type-2 fuzzy-model-based control systems', *Kybernetika*, **50(1)**, 46–65, (2014).