24th European Conference on Artificial Intelligence - ECAI 2020
Santiago de Compostela, Spain

Modelling Belief-Revision Functions at Extended Languages

Theofanis Aravanis and **Pavlos Peppas** and **Mary-Anne Williams**[1]

Abstract. The policy of rational belief revision is encoded in the so-called AGM revision functions. Such functions are characterized (both axiomatically and constructively) within the well-known AGM paradigm, proposed by Alchourrón, Gärdenfors and Makinson. In this article, we show that — although not in a straightforward way — a sufficient extension of the underlying language allows for the modelling of any AGM revision function (defined at the initial language), by means of a Hamming-based rule for belief revision introduced by Dalal (defined at the extended language). The established results enrich the applicability of Dalal's proposal, leading to a conceptual and ontological reduction, as well as open new doors for the construction of any type of revision function in a practical context, given the intuitive appeal and simplicity of Dalal's construction.

1 INTRODUCTION

Belief Revision is the study of knowledge dynamics [9, 8]. This research field was established by the seminal work of Alchourrón, Gärdenfors and Makinson [1], from which the so-called *AGM paradigm* arose, a popular framework modelling the belief-revision process. In the AGM paradigm, a *belief set* (i.e., the agent's belief corpus) is represented as a logical theory, the new epistemic input is represented as a logical sentence, and the process of belief revision is encoded as a function (called *revision function*) that maps a theory and a sentence to a revised (new) theory.

The class of *rational* revision functions, called *AGM revision functions*, is characterized both *axiomatically* and *constructively*. The most popular axiomatic model is a set of eight postulates, known as the *AGM postulates for revision*, that every AGM revision function ought to satisfy. Given that information is valuable and unnecessary loss must be avoided, the underlying motivation for the aforementioned postulates has been the principle of *minimal change*. On the other hand, the most popular constructive model that uniquely defines AGM revision functions is based on a special kind of total preorders over possible worlds, called *faithful preorders* [11].

Several "off-the-shelf" revision operators have been proposed in the literature, implementing the revision process [6, 5, 21, 19, 20]. A popular proposal is that of Dalal [6]. Dalal's revision operator is a specific AGM revision function, based on *Hamming distance*, with numerous appealing features that make it well-suited for implementations. The following list summarizes the most important:

- It is simple and intuitive.
- It is the *only one* among other well-known "off-the-shelf" revision operators (like Borgida's [5], Winslett's [21], Satoh's [19], Weber's [20]) that satisfies the full set of AGM postulates for revision [11, p. 269].
- It has *zero representational cost*, as it can be constructed (via the rule/condition (D) of Section 4) with no extra information concerning the underlying revision policy.
- It is *relevance-sensitive*, since it satisfies Parikh's axiom for relevance in belief revision [14, 18].
- It can cover a wide range of practical belief-revision scenarios, as it is quite natural.

In this article, we show that — although not in a straightforward way — a sufficient *extension* of the underlying (propositional) language allows for the modelling of *any* AGM revision function (defined at the initial language), by means of Dalal's rule (defined at the extended language). That is to say, the added expressivity of the language results in an augmented capability of modelling revision policies.

As the implementation of both *general* and *efficient* belief-revision systems for solving real-world problems has proven to be difficult (due to well-known computational complexity results [7, 12]), the research focuses mainly to the implementation of concrete, compactly-specified revision operators, utilizing a fixed algorithm for encoding the underlying revision policy. One such approach has been, very recently, presented in [10], where an efficient solver for a natural *generalization* of Dalal's revision operator (specifically, a form of *parametrized difference* revision) was developed.[2] Following this line of research, the results of this paper enrich the applicability of Dalal's operator — an AGM revision function perfectly-suited for implementations — and bring the handling of any type of revision function by tools like that of [10] one step closer.

Last but not least, from a theoretical viewpoint, proving that any AGM revision function can be modelled by (a modification of) the well-behaved Dalal's operator leads to a *conceptual and ontological simplification*.

It is worth-noting that an analogous attempt of extending the underlying language has been made in [13], where the authors increase the range of applicability of the notion of minimal change, as part of a broader study concerning the relation of the latter with *causality*.

The rest of the article is structured as follows: The next section introduces basic notation and terminology. Sections 3 and 4 present the AGM paradigm and Dalal's revision operator, respectively. Section 5 establishes an impossibility result that refrains us from a straightforward way of modelling AGM revision functions (defined at the initial language), by means of Dalal's operator (defined at an extended language). As it turns out, an alternative approach is possible for such

[1] University of Patras, Greece, and University of Technology Sydney, Australia, emails: taravanis@upatras.gr, pavlos@upatras.gr, Mary-Anne.Williams@uts.edu.au

[2] For the notion of *parametrized difference* revision, the interested reader is referred to [16, 17, 4].

a modelling, described in Section 6. The last section of the article is devoted to some concluding remarks.

2 FORMAL PRELIMINARIES

This section fixes basic notation and terminology that shall be used throughout the article.

2.1 Language

For a *finite*, non-empty and non-singleton set of propositional variables $\mathcal{P}$, we define $\mathcal{L}$ to be the propositional language generated from $\mathcal{P}$, using the standard Boolean connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), $\leftrightarrow$ (equivalence), $\neg$ (negation), the special symbol $\perp$ (arbitrary contradiction), and governed by *classical propositional logic*.

For a set of sentences Γ of $\mathcal{L}$, $Cn(\Gamma)$ denotes the set of all logical consequences of Γ. For any two sentences $\varphi, \psi \in \mathcal{L}$, we shall write $\varphi \equiv \psi$ iff $Cn(\{\varphi\}) = Cn(\{\psi\})$.

The inference relation $\models$ is used in the following sense:

$$\Gamma \models \varphi \quad \text{iff} \quad \varphi \in Cn(\Gamma).$$

In a similar vein, for any two sets of sentences Γ, Γ' and any two sentences $\varphi, \psi \in \mathcal{L}$, we write $\Gamma \models \Gamma'$ whenever $\Gamma' \subseteq Cn(\Gamma)$, $\psi \models \varphi$ whenever $\varphi \in Cn(\{\psi\})$, and $\models \varphi$ as an alternative notation for $\varphi \in Cn(\varnothing)$.

2.2 Belief Sets

An agent's set of beliefs will be modelled as a *theory*, also referred to as a *belief set*. A theory K of $\mathcal{L}$ is any set of sentences of $\mathcal{L}$ closed under logical consequence; in symbols, $K = Cn(K)$. We denote the set of all consistent theories of $\mathcal{L}$ by $\mathbb{K}$.

A theory K is *complete* iff, for all sentences $\varphi \in \mathcal{L}$, either $\varphi \in K$ or $\neg\varphi \in K$. For a theory K and a set of sentences Γ of $\mathcal{L}$, by $K + \Gamma$ is denoted the logical closure of the set $K \cup \Gamma$, i.e., $Cn(K \cup \Gamma)$, whereas, for a sentence $\varphi \in \mathcal{L}$, $K + \varphi$ abbreviates $K + \{\varphi\}$.

2.3 Literals and Possible Worlds

A *literal* is a propositional variable $p \in \mathcal{P}$ or its negation. We define a *possible world* (or simply a *world*) r to be a consistent set of literals, such that, for any propositional variable $p \in \mathcal{P}$, either $p \in r$ or $\neg p \in r$. The set of all possible worlds is denoted by $\mathbb{M}$.

For a sentence (set of sentences) φ of $\mathcal{L}$, $[\varphi]$ is the set of worlds at which φ is true. For a possible world r of $\mathbb{M}$, $\bigwedge r$ denotes the conjunction of the literals in r; clearly, $\bigwedge r$ is a sentence of $\mathcal{L}$. For the sake of readability, the negation of a propositional variable p will, sometimes, be represented as $\overline{p}$, instead of $\neg p$. Moreover, possible worlds will, occasionally, be represented as sequences (rather than sets) of literals.

2.4 Preorders

A *preorder* over a set V is any reflexive, transitive binary relation in V. The preorder $\preceq$ is called *total* iff, for all $r, r' \in V$, $r \preceq r'$ or $r' \preceq r$. As usual, we shall denote by $\prec$ the strict part of $\preceq$; i.e., $r \prec r'$ iff $r \preceq r'$ and $r' \not\preceq r$. Moreover, for any $X \subseteq V$, by $min(X, \preceq)$ we denote the set of the minimal elements of X, with respect to $\preceq$; in symbols,

$$min(X, \preceq) = \Big\{ r \in X : \text{for all } r' \in X, \text{if } r' \preceq r, \text{then } r \preceq r' \Big\}.$$

The next definition introduces the notion of layers of a preorder over possible worlds.

Definition 1 (Preorder's Layers) *Let $\preceq$ be a total preorder over the possible worlds of $\mathbb{M}$. Consider the sequence of sets of worlds L_1, L_2, ..., defined recursively as follows:*

$$\begin{aligned} L_1 &= min(\mathbb{M}, \preceq) \\ L_2 &= min(\mathbb{M} - L_1, \preceq) \\ &\vdots \\ L_{j+1} &= min\Big(\mathbb{M} - (\bigcup_{i=1}^{j} L_i), \preceq\Big) \\ &\vdots \end{aligned}$$

Clearly, since $\mathbb{M}$ is finite, at some point the above sequence will reach the empty set, and will remain equal to the empty set from then onwards. Let us denote by k the index of the last non-empty set in the above sequence; i.e., $L_k \neq \varnothing$ and $L_{k+i} = \varnothing$, for all $i \geqslant 1$. Essentially, $L_1, L_2, \ldots, L_k$ are the equivalence classes induced from $\preceq$, and the set $L = \{L_1, L_2, \ldots, L_k\}$ is a partition of $\mathbb{M}$.

We shall refer to the set of worlds L_1 as the first layer of $\preceq$, to L_2 as the second layer of $\preceq$, and so forth.

Given a fixed set of worlds $\mathbb{M}$, the "longest" preorder (namely, the preorder with the maximum number of layers) is a preorder such that $|L_1| = |L_2| = \ldots = |L_k| = 1$.[3]

2.5 Extended Languages

A propositional language $\mathcal{L}^+$ is called an *extension* of $\mathcal{L}$ iff $\mathcal{L}^+$ is finitary, and, moreover, it is built from a set of propositional variables $\mathcal{P}^+$, such that $\mathcal{P} \subset \mathcal{P}^+$. Accordingly, $\mathbb{M}^+$ denotes the set of all possible worlds defined over $\mathcal{L}^+$ (or $\mathcal{P}^+$). Notation $\overline{\mathcal{L}}$ denotes the complement language of $\mathcal{L}$; that is, the language built from the propositional variables of the set $\mathcal{P}^+ - \mathcal{P}$.

For a possible world r of $\mathbb{M}^+$, $r_{\mathcal{L}}$ denotes the *restriction* of r in $\mathcal{L}$ (i.e., $r_{\mathcal{L}} = r \cap \mathcal{L}$); clearly, $r_{\mathcal{L}} \in \mathbb{M}$. Lastly, for a theory K of $\mathcal{L}$, we shall call an *extension* of K in $\mathcal{L}^+$, denoted by $K^\diamond$, any theory of $\mathcal{L}^+$ such that $K^\diamond \cap \mathcal{L} = K$.

3 THE AGM PARADIGM

In this section, the *axiomatic* approach of the AGM paradigm is briefly discussed, as well as a popular *constructive model* for this process, based on a special kind of total preorders over possible worlds, called *faithful preorders*.

3.1 The AGM Postulates for Revision

In the AGM paradigm, the process of belief revision is modelled as a (binary) function $*$, mapping a theory K and a sentence φ to the revised (new) theory $K * \varphi$; i.e., $* : \mathbb{K} \times \mathcal{L} \mapsto \mathbb{K}$. The *AGM postulates for revision* $(K * 1)$–$(K * 8)$, presented subsequently, circumscribe the territory of *rational* revision functions, the so-called *AGM revision functions*.[4]

---

[3] For a set V, $|V|$ denotes the cardinality of V.
[4] See [9] or [15] for a detailed elaboration on the postulates.

($\mathbf{K*1}$) $K*\varphi$ is a theory of $\mathcal{L}$.

($\mathbf{K*2}$) $\varphi \in K*\varphi$.

($\mathbf{K*3}$) $K*\varphi \subseteq K+\varphi$.

($\mathbf{K*4}$) If $\neg\varphi \notin K$, then $K+\varphi \subseteq K*\varphi$.

($\mathbf{K*5}$) $K*\varphi$ is inconsistent iff $\models \neg\varphi$.

($\mathbf{K*6}$) If $\varphi \equiv \psi$, then $K*\varphi = K*\psi$.

($\mathbf{K*7}$) $K*(\varphi\wedge\psi) \subseteq (K*\varphi)+\psi$.

($\mathbf{K*8}$) If $\neg\psi \notin K*\varphi$, then $(K*\varphi)+\psi \subseteq K*(\varphi\wedge\psi)$.

The underlying motivation for postulates $(K*1)$–$(K*8)$ is the principle of *minimal change*, according to which a rational agent ought to change her/his beliefs *as little as possible* in order to (consistently) accommodate the new information.

The AGM postulates for revision correspond to the logical properties that any rational revision function has to possess, and they do not *uniquely* determine a specific AGM revision function. A *constructive* way to define AGM revision functions is presented in the remainder of this section.

3.2 Faithful-Preorders Model

There are several methods to define the revision process of a theory K by a sentence φ, but, in principle, they are all equivalent to the *faithful-preorders model*, proposed by Katsuno and Mendelzon [11]. In particular, Katsuno and Mendelzon showed that any AGM revision function can be *constructed/defined* with the use of *faithful preorders*, a special kind of total preorders over all possible worlds.

Definition 2 (Faithful Preorder, [11]) *For a theory K of $\mathcal{L}$, a preorder over possible worlds $\preceq_K$ is faithful to K iff it is total, and such that, for any possible worlds $r, r' \in \mathbb{M}$:*[5]

- **(i)** *If $r \in [K]$, then $r \preceq_K r'$.*
- **(ii)** *If $r \in [K]$ and $r' \notin [K]$, then $r \prec_K r'$.*

The above definition entails that the $\preceq_K$-minimal worlds define the belief set K to which the preorder $\preceq_K$ is faithful; i.e., $[K] = min(\mathbb{M}, \preceq_K)$.

Definition 3 (Faithful Assignment, [11]) *A faithful assignment is a function that maps each theory K of $\mathcal{L}$ to a preorder $\preceq_K$ over the possible worlds of $\mathbb{M}$, that is faithful to K.*

The following representation result has been established by Katsuno and Mendelzon.

Theorem 1 ([11]) *A revision operator $*$ satisfies postulates $(K*1)$–$(K*8)$ iff there exists a faithful assignment such that, for every $K \in \mathbb{K}$ and $\varphi \in \mathcal{L}$:*

$$\textbf{(F*)} \quad [K*\varphi] = min([\varphi], \preceq_K).$$

From an epistemological point of view, a faithful preorder $\preceq_K$ encodes the *comparative plausibility* of the possible worlds of $\mathbb{M}$, relative to K, with the more plausible worlds appearing lower in the ordering. Hence, condition (F$*$) defines the revised belief set $K*\varphi$ as the theory corresponding to the most plausible (with respect to K) φ-worlds.

For ease of presentation, throughout this article, we shall consider only *consistent* belief sets, and *contingent* epistemic input.

[5] In [11], faithful preorders are associated with sentences, rather than theories. Nevertheless, since we deal with a language built over a finite set of propositional variables, the difference is immaterial.

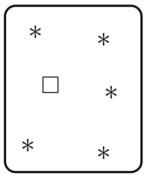

Figure 1. The class of AGM revision functions.

4 DALAL'S REVISION OPERATOR

A very natural way of defining the preorder $\preceq_K$, associated to a theory K, has been proposed by Dalal [6]. Dalal defines the plausibility of worlds, encoded in $\preceq_K$, in terms of a *Hamming-based difference* between worlds. Before presenting Dalal's proposal, let us introduce the necessary definitions.

Definition 4 (Difference Between Worlds) *The difference between two worlds w, r of $\mathbb{M}$, denoted by Diff (w, r), is the set of propositional variables over which the two worlds disagree. In symbols:*

$$\mathit{Diff}\,(w, r) = \Big((w-r) \cup (r-w)\Big) \cap \mathcal{P}.$$

Definition 5 (Distance Between Theories and Worlds, [6]) *The distance between a theory K of $\mathcal{L}$ and a world r of $\mathbb{M}$, denoted by Dist (K, r), is as follows:*

$$\mathit{Dist}\,(K, r) = \min_{w\in[K]} \Big(\big|\mathit{Diff}\,(w, r)\big|\Big).$$

Definition 6 (Dalal's Operator, [6]) *Dalal's operator, denoted by $\Box$, is the revision function induced from the following (Dalal's) preorder, denoted by $\sqsubseteq_K$ and associated with a theory K of $\mathcal{L}$, via condition (F$*$):*

$$\textbf{(D)} \quad r \sqsubseteq_K r' \quad \textit{iff} \quad \mathit{Dist}\,(K, r) \leqslant \mathit{Dist}\,(K, r').$$

It can be easily shown that, for each theory K of $\mathcal{L}$, $\sqsubseteq_K$ is a total preorder faithful to K, as well as that $\Box$ is an AGM revision function [11, p. 269].

Figure 1 depicts Dalal's revision operator $\Box$ relative to the whole class of AGM revision functions.

Dalal's proposal, essentially, constitutes an *algorithm* or *rule* that can be used in order to implement the revision policy encoded in $\Box$, when no information is available (from the agent) about the relative plausibility of possible worlds.

Adopting Dalal's rule, the preorder assigned at a theory K of $\mathcal{L}$ is *fixed* and it is determined via condition (D). As a consequence, there exist total preorders over possible worlds, for which Dalal prohibits their association with K. This, in turn, constitutes a barrier for modelling an *arbitrary* AGM revision function by means of Dalal's approach.

As it will be shown subsequently, a sufficient *extension* of the underlying language remedies this weakness, although *not in a straightforward way*, as the impossibility result of the next section implies.

5 AN IMPOSSIBILITY RESULT

In this section, we point out an impossibility result that refrains us from a straightforward way of modelling an arbitrary AGM revision function, defined at the initial language $\mathcal{L}$, by means of Dalal's operator, defined at an extension of $\mathcal{L}$. First however, we need some preliminary material.

Definition 7 (Revision-Equivalent Operators) *Let $*$ be an AGM revision function defined at $\mathcal{L}$, and let $\square$ be Dalal's revision operator defined at an extension $\mathcal{L}^+$ of $\mathcal{L}$. We shall say that $*$ and $\square$ are revision-equivalent modulo $\mathcal{L}$ iff, for any theory K of $\mathcal{L}$, any extension $K^\diamond$ of K in $\mathcal{L}^+$, and all $\varphi \in \mathcal{L}$, $K * \varphi = (K^\diamond \square\, \varphi) \cap \mathcal{L}$.*

Now, we shall introduce an interesting type of sublanguage-projection of a faithful preorder, called *filtering*. The notion of filtering has been proposed in [18], for the formulation of the faithful-preorders characterization of Parikh's relevance sensitive axiom for belief revision [14].[6]

Definition 8 (Faithful-Preorder Filtering, [18]) *Let $\mathcal{L}^+$ be an extension of $\mathcal{L}$. For a total preorder $\preceq$ over the possible worlds of $\mathbb{M}^+$, the $\mathcal{L}$-filtering of $\preceq$, denoted by ${}^{\mathcal{L}}\!\preceq$, is a total preorder over the possible worlds of $\mathbb{M}$ defined as follows:*[7]

$$r_{\mathcal{L}} \; {}^{\mathcal{L}}\!\preceq r'_{\mathcal{L}} \quad \textit{iff} \quad \begin{array}{c} \textit{for some } w \in min([r_{\mathcal{L}}], \preceq) \\ \textit{and some } w' \in min([r'_{\mathcal{L}}], \preceq), \end{array} \quad w \preceq w'.$$

In the above definition, the worlds $r_{\mathcal{L}}$, $r'_{\mathcal{L}}$ in $r_{\mathcal{L}} \; {}^{\mathcal{L}}\!\preceq r'_{\mathcal{L}}$ belong to $\mathbb{M}$ (as the preorder ${}^{\mathcal{L}}\!\preceq$ is defined over $\mathbb{M}$), whereas, the sets of worlds $[r_{\mathcal{L}}]$, $[r'_{\mathcal{L}}]$ in $min([r_{\mathcal{L}}], \preceq)$, $min([r'_{\mathcal{L}}], \preceq)$, respectively, are subsets of $\mathbb{M}^+$ (as the preorder $\preceq$ is defined over $\mathbb{M}^+$); analogous observations apply in any similar cases in what follows.

It is not hard to verify that, if the preorder ${}^{\mathcal{L}}\!\preceq$ is faithful to a theory K of $\mathcal{L}$, then the initial preorder $\preceq$ is faithful to an extension $K^\diamond$ of K in $\mathcal{L}^+$. Moreover, if $\mathcal{L} = \mathcal{L}^+$, then ${}^{\mathcal{L}}\!\preceq \, = \, \preceq$.

To understand the intuition behind filtering, a concrete example is presented subsequently.

Example 1 *Suppose that $\mathcal{L}$ is built from $\mathcal{P} = \{a, b\}$, and that an extension $\mathcal{L}^+$ of $\mathcal{L}$ is built from $\mathcal{P}^+ = \{a, b, c\}$. Moreover, let K be a theory of $\mathcal{L}$, such that $K = Cn\big(\{a \leftrightarrow b\}\big)$, and let $K^\diamond$ be an extension of K in $\mathcal{L}^+$, such that $K^\diamond = Cn\big(\{a \leftrightarrow b, c\}\big)$. Then, the only Dalal's preorder $\sqsubseteq_{K^\diamond}$ that is assigned at $K^\diamond$ is the following:*[8]

$$\begin{array}{c} abc \\ \overline{a}\overline{b}c \end{array} \quad \sqsubset_{K^\diamond} \quad \begin{array}{c} \overline{a}bc \\ a\overline{b}c \\ ab\overline{c} \\ \overline{a}\overline{b}\overline{c} \end{array} \quad \sqsubset_{K^\diamond} \quad \begin{array}{c} a\overline{b}\overline{c} \\ \overline{a}b\overline{c} \end{array}$$

The $\mathcal{L}$-filtering of $\sqsubseteq_{K^\diamond}$ is as follows:

$$\begin{array}{c} ab \\ \overline{a}\overline{b} \end{array} \quad {}^{\mathcal{L}}\!\sqsubset_{K^\diamond} \quad \begin{array}{c} \overline{a}b \\ a\overline{b} \end{array}$$

Observe that the preorder ${}^{\mathcal{L}}\!\sqsubseteq_{K^\diamond}$ is associated with theory K of $\mathcal{L}$, and that it is a Dalal's preorder as well (see Lemma 2).

[6] For the characterization of Parikh's axiom in the realm of other well-known constructive models, the interested reader is referred to [2, 3].

[7] The definition for filtering used herein is slightly different from that used in [18], but the two are clearly equivalent, as the world w' is *minimal* with respect to $\preceq$.

[8] As usual, for a preorder $\sqsubseteq$, $\sqsubset$ denotes the strict part of $\sqsubseteq$.

Thereafter, consider the following two lemmas.

Lemma 1 *Let $*$ be an AGM revision function defined at $\mathcal{L}$, and let $\square$ be Dalal's revision operator defined at an extension $\mathcal{L}^+$ of $\mathcal{L}$. Moreover, let K be any theory of $\mathcal{L}$, and let $\preceq_K$ be the faithful preorder corresponding to $*$, by means of (F$*$). If $*$ and $\square$ are revision-equivalent modulo $\mathcal{L}$, then $\preceq_K$ is identical to the $\mathcal{L}$-filtering of the Dalal's preorder $\sqsubseteq_{K^\diamond}$, associated with any extension $K^\diamond$ of K in $\mathcal{L}^+$; i.e., $\preceq_K \, = \, {}^{\mathcal{L}}\!\sqsubseteq_{K^\diamond}$.*

Proof. Assume that $*$ and $\square$ are revision-equivalent modulo $\mathcal{L}$. Therefore, for any theory K of $\mathcal{L}$, any extension $K^\diamond$ of K in $\mathcal{L}^+$, and all $\varphi \in \mathcal{L}$, $K * \varphi = (K^\diamond \square\, \varphi) \cap \mathcal{L}$. This again entails that $min([\varphi], \preceq_K) = \big\{u_{\mathcal{L}} : u \in min([\varphi], \sqsubseteq_{K^\diamond})\big\}$.[9] First, we show that $\preceq_K \, \subseteq \, {}^{\mathcal{L}}\!\sqsubseteq_{K^\diamond}$.

Consider any two worlds $r, r' \in \mathbb{M}$, such that $r \preceq_K r'$. Construct the sentence φ as follows: $\varphi = \big(\bigwedge r\big) \vee \big(\bigwedge r'\big)$. Clearly, $\varphi \in \mathcal{L}$, $[\varphi] = \{r, r'\}$, and thus $r \in min([\varphi], \preceq_K)$. From $min([\varphi], \preceq_K) = \big\{u_{\mathcal{L}} : u \in min([\varphi], \sqsubseteq_{K^\diamond})\big\}$, it follows that $r \in \big\{u_{\mathcal{L}} : u \in min([\varphi], \sqsubseteq_{K^\diamond})\big\}$. In other words, $w \sqsubseteq_{K^\diamond} w'$ for some $w \in min([r], \sqsubseteq_{K^\diamond})$, and some $w' \in min([r'], \sqsubseteq_{K^\diamond})$; hence, $r \; {}^{\mathcal{L}}\!\sqsubseteq_{K^\diamond} r'$. Consequently, $\preceq_K \, \subseteq \, {}^{\mathcal{L}}\!\sqsubseteq_{K^\diamond}$, as desired.

The proof of the converse, i.e., ${}^{\mathcal{L}}\!\sqsubseteq_{K^\diamond} \, \subseteq \, \preceq_K$, is totally symmetric, as the reader can easily verify. ∎

Lemma 2 *Let $\mathcal{L}^+$ be an extension of $\mathcal{L}$, and let $\preceq$ be a total preorder over the possible worlds $\mathbb{M}^+$. If $\preceq$ is a Dalal's preorder, then the $\mathcal{L}$-filtering of $\preceq$, i.e., ${}^{\mathcal{L}}\!\preceq$, is a Dalal's preorder.*

Proof. Call K the belief set that the preorder ${}^{\mathcal{L}}\!\preceq$ is faithful to. As already stated, the preorder $\preceq$ is faithful to an extension $K^\diamond$ of K in $\mathcal{L}^+$.

Suppose that $\preceq$ is a Dalal's preorder; namely, according to condition (D), $r \preceq r'$ iff *Dist* $(K^\diamond, r) \leqslant$ *Dist* $(K^\diamond, r')$, for all $r, r' \in \mathbb{M}^+$. To prove that ${}^{\mathcal{L}}\!\preceq$ is a Dalal's preorder as well, it suffices to show that $r_{\mathcal{L}} \; {}^{\mathcal{L}}\!\preceq r'_{\mathcal{L}}$ iff *Dist* $(K, r_{\mathcal{L}}) \leqslant$ *Dist* $(K, r'_{\mathcal{L}})$.[10]

Let r, r' be any two worlds of $\mathbb{M}^+$. First, observe that $\big\{u \cap \overline{\mathcal{L}} : u \in [r_{\mathcal{L}}]\big\} = \big\{u' \cap \overline{\mathcal{L}} : u' \in [r'_{\mathcal{L}}]\big\}$. Therefore, it is not hard to verify that:

$$\textit{Dist}\,(K, r_{\mathcal{L}}) \leqslant \textit{Dist}\,(K, r'_{\mathcal{L}}) \quad \text{iff} \quad \begin{array}{r} \textit{Dist}\,(K^\diamond, w) \leqslant \textit{Dist}\,(K^\diamond, w'), \\ \text{where } w \in min([r_{\mathcal{L}}], \preceq), \\ \text{and } w' \in min([r'_{\mathcal{L}}], \preceq). \end{array}$$

Let us call the aforementioned condition (1).

Now, assume that $r_{\mathcal{L}} \; {}^{\mathcal{L}}\!\preceq r'_{\mathcal{L}}$. By the definition of ${}^{\mathcal{L}}\!\preceq$, we have that $w \preceq w'$, where $w \in min([r_{\mathcal{L}}], \preceq)$ and $w' \in min([r'_{\mathcal{L}}], \preceq)$. Hence from condition (D), *Dist* $(K^\diamond, w) \leqslant$ *Dist* $(K^\diamond, w')$, where $w \in min([r_{\mathcal{L}}], \preceq)$ and $w' \in min([r'_{\mathcal{L}}], \preceq)$. Therefore from (1), it follows that *Dist* $(K, r_{\mathcal{L}}) \leqslant$ *Dist* $(K, r'_{\mathcal{L}})$.

Next, assume that *Dist* $(K, r_{\mathcal{L}}) \leqslant$ *Dist* $(K, r'_{\mathcal{L}})$. Hence from (1), we derive that *Dist* $(K^\diamond, w) \leqslant$ *Dist* $(K^\diamond, w')$, where $w \in min([r_{\mathcal{L}}], \preceq)$ and $w' \in min([r'_{\mathcal{L}}], \preceq)$. Thus from condition (D), $w \preceq w'$, where $w \in min([r_{\mathcal{L}}], \preceq)$ and $w' \in min([r'_{\mathcal{L}}], \preceq)$. Therefore, by the definition of ${}^{\mathcal{L}}\!\preceq$, it follows that $r_{\mathcal{L}} \; {}^{\mathcal{L}}\!\preceq r'_{\mathcal{L}}$.

Consequently, we have shown that $r_{\mathcal{L}} \; {}^{\mathcal{L}}\!\preceq r'_{\mathcal{L}}$ iff *Dist* $(K, r_{\mathcal{L}}) \leqslant$ *Dist* $(K, r'_{\mathcal{L}})$, as desired. ∎

[9] As earlier stated, $[\varphi]$ in $min([\varphi], \preceq_K)$ is a subset of $\mathbb{M}$ (since the preorder $\preceq_K$ is defined over $\mathbb{M}$), whereas, $[\varphi]$ in $min([\varphi], \sqsubseteq_{K^\diamond})$ is a subset of $\mathbb{M}^+$ (since the preorder $\sqsubseteq_{K^\diamond}$ is defined over $\mathbb{M}^+$).

[10] Note that *Dist* $(K, r) =$ *Dist* $(K, r_{\mathcal{L}})$, as theory K is a subset of $\mathcal{L}$.

Having established the required preliminaries, we proceed to the alluded impossibility result.

Theorem 2 *Let $*$ be any non-Dalal's AGM revision function defined at $\mathcal{L}$. There is no Dalal's operator defined at any extension of $\mathcal{L}$, such that it is revision-equivalent modulo $\mathcal{L}$ to $*$.*

Proof. Firstly, observe that there exists a *non-Dalal's* preorder $\preceq_K$, associated with a theory K of $\mathcal{L}$, that corresponds to $*$, by means of (F$*$).[11]

Suppose, towards contradiction, that there exists a Dalal's operator $\square$ defined at an extension $\mathcal{L}^+$ of $\mathcal{L}$, such that it is revision-equivalent modulo $\mathcal{L}$ to $*$. Therefore, from Lemma 1, we derive that $\preceq_K = {}^{\mathcal{L}}\sqsubseteq_{K^\diamond}$, where $\sqsubseteq_{K^\diamond}$ is the Dalal's preorder associated with any extension $K^\diamond$ of K in $\mathcal{L}^+$. However, from Lemma 2, it follows that the preorder ${}^{\mathcal{L}}\sqsubseteq_{K^\diamond}$ is a Dalal's preorder. Contradiction. ■

6 AN ALTERNATIVE APPROACH FOR MODELLING

Theorem 2 constitutes an impasse for modelling AGM revision functions by means of Dalal's proposal, via revision-equivalence of Definition 7. This section is devoted to the presentation of an alternative approach for such modelling.

6.1 Initial Considerations

Definition 9 (Derivative) *A derivative of $\mathbb{M}$ at an extension $\mathcal{L}^+$ of $\mathcal{L}$, denoted by $\mathbb{M}^\diamond$, is a minimal set of worlds of $\mathbb{M}^+$, such that $\{r \cap \mathcal{L} : \text{for all } r \in \mathbb{M}^\diamond\} = \mathbb{M}$.*

Essentially, the set of worlds resulting from the restriction of all worlds of $\mathbb{M}^\diamond$ in $\mathcal{L}$ coincides with the set $\mathbb{M}$. Since $\mathbb{M}^\diamond$ is minimally defined, it holds that $|\mathbb{M}^\diamond| = |\mathbb{M}|$. Moreover, given any theory K of $\mathcal{L}$, it follows from the above definition that $\{r \cap \mathcal{L} : \text{for some worlds } r \in \mathbb{M}^\diamond\} = [K]$, for any derivative $\mathbb{M}^\diamond$ of $\mathbb{M}$ at $\mathcal{L}^+$.

Example 2 *Suppose that $\mathcal{L}$ is built from $\mathcal{P} = \{a, b\}$, and an extension $\mathcal{L}^+$ of $\mathcal{L}$ is built from $\mathcal{P}^+ = \{a, b, c\}$. Then clearly, $\mathbb{M} = \{ab, a\bar{b}, \bar{a}b, \bar{a}\bar{b}\}$, and a derivative $\mathbb{M}^\diamond$ of $\mathbb{M}$ at $\mathcal{L}^+$ is the set $\{ab\bar{c}, a\bar{b}c, \bar{a}bc, \bar{a}\bar{b}\bar{c}\}$. Firstly, observe that $|\mathbb{M}^\diamond| = |\mathbb{M}|$. If we consider now a theory K of $\mathcal{L}$, such that $K = Cn(\{a \leftrightarrow b\})$, we have that $\{ab\bar{c} \cap \mathcal{L}\ ,\ \bar{a}\bar{b}\bar{c} \cap \mathcal{L}\} = [K]$.*

Intuitively, a derivative $\mathbb{M}^\diamond$ can be regarded as some "valid" portion of $\mathbb{M}^+$, determined by a set of *integrity constraints*.

Definition 10 (Restricted Dalal's Preorder) *Let K be a theory of $\mathcal{L}$, and $\mathbb{M}^\diamond$ be a derivative of $\mathbb{M}$ at an extension $\mathcal{L}^+$ of $\mathcal{L}$. We shall denote by $\sqsubseteq_{\langle \mathbb{M}^\diamond, K^\diamond\rangle}$ the restricted Dalal's preorder over $\mathbb{M}^\diamond$, associated with an extension $K^\diamond$ of K in $\mathcal{L}^+$, such that $\Big\{r \cap \mathcal{L} : r \in min\big(\mathbb{M}^\diamond, \sqsubseteq_{\langle \mathbb{M}^\diamond, K^\diamond\rangle}\big)\Big\} = [K]$.*

The preorder $\sqsubseteq_{\langle \mathbb{M}^\diamond, K^\diamond\rangle}$ is called restricted as it constitutes a *restriction* (to $\mathbb{M}^\diamond$) of the Dalal's preorder associated with theory $K^\diamond$ and specified over $\mathbb{M}^+$; hence, it is a total preorder over $\mathbb{M}^\diamond$. Notice, moreover, that the derivative $\mathbb{M}^\diamond$, along with theory K, *uniquely* define theory $K^\diamond$.

[11] Recall that the set of propositional variables $\mathcal{P}$ is a non-singleton set; hence, such a non-Dalal's preorder indeed exists.

Definition 11 (Preorder Projection) *Let $\mathcal{L}^+$ be an extension of $\mathcal{L}$, $\mathbb{M}^\diamond$ be a derivative of $\mathbb{M}$ at $\mathcal{L}^+$, and let $\preceq$ be a total preorder over $\mathbb{M}^\diamond$. The projection of $\preceq$ to $\mathcal{L}$, denoted by $\preceq^{\mathcal{L}}$, is an ordering over $\mathbb{M}$, such that, for all $w, w' \in \mathbb{M}$:*

$$w \preceq^{\mathcal{L}} w' \quad \textit{iff} \quad r \preceq r', \quad \begin{array}{c}\textit{for some } r, r' \in \mathbb{M}^\diamond, \textit{ such that} \\ w = r_{\mathcal{L}} \textit{ and } w' = r'_{\mathcal{L}}.\end{array}$$[12]

By the definition of a derivative, it is not hard to verify that $\preceq^{\mathcal{L}}$ is a *total preorder* over $\mathbb{M}$. The intuition behind Definitions 10 and 11 is illustrated in Example 3 of the next subsection.

6.2 Augmenting Dalal's Proposal

It turns out that the added expressivity of an extension of $\mathcal{L}$ results in an augmented capability of Dalal's proposal. The following example is illustrative.

Example 3 *Suppose that $\mathcal{L}$ is built from $\mathcal{P} = \{a, b\}$. Moreover, let K be a (complete) theory of $\mathcal{L}$, such that $K = Cn(\{a, b\})$. Then, the only Dalal's preorder $\sqsubseteq_K$ that is assigned at K is the following:*

$$ab \quad \sqsubset_K \quad \begin{array}{c} a\bar{b} \\ \bar{a}b \end{array} \quad \sqsubset_K \quad \bar{a}\bar{b}$$

Suppose that we want to assign at theory K the following non-Dalal's preorder $\preceq_K$:

$$ab \quad \prec_K \quad \bar{a}b \quad \prec_K \quad a\bar{b} \quad \prec_K \quad \bar{a}\bar{b}$$

To this end, consider the derivative $\mathbb{M}^\diamond = \{abc, \bar{a}bc, a\bar{b}\bar{c}, \bar{a}\bar{b}\bar{c}\}$. Then, the restricted Dalal's preorder $\sqsubseteq_{\langle \mathbb{M}^\diamond, K^\diamond\rangle}$, associated with the extension $K^\diamond = Cn(\{a, b, c\})$ of K in $\mathcal{L}^+$, is as follows:

$$abc \quad \sqsubset_{\langle \mathbb{M}^\diamond, K^\diamond\rangle} \quad \bar{a}bc \quad \sqsubset_{\langle \mathbb{M}^\diamond, K^\diamond\rangle} \quad a\bar{b}\bar{c} \quad \sqsubset_{\langle \mathbb{M}^\diamond, K^\diamond\rangle} \quad \bar{a}\bar{b}\bar{c}$$

Clearly, the projection of $\sqsubseteq_{\langle \mathbb{M}^\diamond, K^\diamond\rangle}$ to $\mathcal{L}$, i.e., $\sqsubseteq^{\mathcal{L}}_{\langle \mathbb{M}^\diamond, K^\diamond\rangle}$, is identical to $\preceq_K$.

Now, suppose that we want to assign at theory K the following non-Dalal's preorder $\preceq'_K$:

$$ab \quad \prec'_K \quad \begin{array}{c} \bar{a}b \\ a\bar{b} \\ \bar{a}\bar{b} \end{array}$$

In this case, the restricted Dalal's preorder $\sqsubseteq_{\langle \mathbb{M}^{\diamond\prime}, K^\diamond\rangle}$ over $\mathbb{M}^{\diamond\prime} = \{abc, \bar{a}b\bar{c}, a\bar{b}\bar{c}, \bar{a}\bar{b}c\}$, associated with $K^\diamond$, produces, after its projection to $\mathcal{L}$, the desired preorder $\preceq'_K$.

Note that both $\sqsubseteq_{\langle \mathbb{M}^\diamond, K^\diamond\rangle}$ and $\sqsubseteq_{\langle \mathbb{M}^{\diamond\prime}, K^\diamond\rangle}$ are restrictions (parts) of the Dalal's preorder, associated with $K^\diamond$ and specified over all possible worlds of $\mathbb{M}^+$, shown below:

$$abc \quad \sqsubset_{K^\diamond} \quad \begin{array}{c} \bar{a}bc \\ a\bar{b}c \\ ab\bar{c} \end{array} \quad \sqsubset_{K^\diamond} \quad \begin{array}{c} \bar{a}\bar{b}c \\ a\bar{b}\bar{c} \\ \bar{a}b\bar{c} \end{array} \quad \sqsubset_{K^\diamond} \quad \bar{a}\bar{b}\bar{c}$$

Theorem 3 formalizes what the above example indirectly suggests.

Theorem 3 *Let K be any theory of $\mathcal{L}$, and let $\preceq_K$ be any total preorder over $\mathbb{M}$, faithful to K. In a sufficiently extended language $\mathcal{L}^+$ of $\mathcal{L}$, there exist a derivative $\mathbb{M}^\diamond$ of $\mathbb{M}$ at $\mathcal{L}^+$ and an extension $K^\diamond$ of K in $\mathcal{L}^+$, such that, for the restricted Dalal's preorder $\sqsubseteq_{\langle \mathbb{M}^\diamond, K^\diamond\rangle}$ over $\mathbb{M}^\diamond$, $\sqsubseteq^{\mathcal{L}}_{\langle \mathbb{M}^\diamond, K^\diamond\rangle} = \preceq_K$.*

[12] Notice that, for every world $w \in \mathbb{M}$, there is only one world $r \in \mathbb{M}^\diamond$, such that $w = r_{\mathcal{L}}$.

Proof. To prove the theorem, we shall start with the worlds of $\mathbb{M}$ and, progressively, construct a derivative $\mathbb{M}^\diamond$ of $\mathbb{M}$ at a sufficient extension $\mathcal{L}^+$ of $\mathcal{L}$, such that, for the restricted Dalal's preorder $\sqsubseteq_{\langle \mathbb{M}^\diamond, K^\diamond \rangle}$ (associated with an extension $K^\diamond$ of K in $\mathcal{L}^+$), its projection to $\mathcal{L}$ is identical to $\preceq_K$. Note that the way that $\mathbb{M}^\diamond$ shall be constructed implies that $\mathcal{L}^+$ is *minimally* defined (with respect to set inclusion). In what follows, r_i denotes an arbitrary world in the i-th layer L_i of the preorder $\preceq_K$, for $1 \leqslant i \leqslant k$, where k is the last layer of $\preceq_K$.

For each world $r \in \mathbb{M}$, construct the (extended) world $z_r = r \cup l_r$, where l_r is a set of the *minimum* number n of literals, induced from propositional variables that do not appear in $\mathcal{P}$, so that the following conditions *jointly* hold:

- **(i)** For any world $r_1 \in \mathbb{M}$, the set l_{r_1} contains *only positive* literals. Thus, the set of (extended) worlds $\{z_{r_1} : \text{for all } r_1 \in \mathbb{M}\}$ defines an extension $K^\diamond$ of K in $\mathcal{L}^+$.
- **(ii)** For any world $r_2 \in \mathbb{M}$, *Dist* $(K^\diamond, z_{r_2}) = \max_{r \in L_2} \big(\textit{Dist}\,(K, r)\big)$. Thus, for all $r_2 \in \mathbb{M}$ such that *Dist* $(K, r_2) = \max_{r \in L_2} \big(\textit{Dist}\,(K, r)\big)$, the set l_{r_2} contains *only positive* literals, whereas, for all $r_2 \in \mathbb{M}$ such that *Dist* $(K, r_2) < \max_{r \in L_2} \big(\textit{Dist}\,(K, r)\big)$, the set l_{r_2} contains both positive and negative literals.
- **(iii)** For any world $r_i \in \mathbb{M}$ such that $3 \leqslant i \leqslant k$, *Dist* $(K^\diamond, z_{r_i}) =$ *Dist* $(K^\diamond, z_{r_{i-1}}) + 1$. Thus, for some $u \in L_k$, the set l_u contains *only negative* literals.

From conditions (i)–(iii), we derive that the set of worlds $\{z_r : \text{for all } r \in \mathbb{M}\}$ is a derivative $\mathbb{M}^\diamond$ of $\mathbb{M}$ at $\mathcal{L}^+$. Moreover, for any $r, r' \in \mathbb{M}$, $r \preceq_K r'$ iff *Dist* $(K^\diamond, z_r) \leqslant$ *Dist* $(K^\diamond, z_{r'})$; thus, by condition (D), it follows that $r \preceq_K r'$ iff $z_r \sqsubseteq_{\langle \mathbb{M}^\diamond, K^\diamond \rangle} z_{r'}$. Lastly, since $\mathbb{M}^\diamond$ is a derivative of $\mathbb{M}$ at $\mathcal{L}^+$, we derive that $\sqsubseteq^{\mathcal{L}}_{\langle \mathbb{M}^\diamond, K^\diamond \rangle} = \preceq_K$, as desired. ■

Informally speaking, Theorem 3 says that, given *any* theory K of the initial language $\mathcal{L}$ and *any* faithful preorder $\preceq_K$ associated with K, we can find a Dalal's preorder, defined at a sufficiently *extended* language $\mathcal{L}^+$ of $\mathcal{L}$, in which the preorder $\preceq_K$ is embedded. Otherwise stated, *any* faithful preorder $\preceq_K$, defined at $\mathcal{L}$, can be "extracted" from a Dalal's preorder, defined at $\mathcal{L}^+$.

Although the language $\mathcal{L}^+$ in the proof of Theorem 3 was minimally constructed (with respect to set inclusion), we have not identified yet the *exact minimum* number of extra propositional variables required to produce an extension $\mathcal{L}^+$ of $\mathcal{L}$, in order for *any* faithful preorder at $\mathcal{L}$ to be modelled by means of a restricted Dalal's preorder at $\mathcal{L}^+$. The following proposition enlightens this aspect.

Proposition 1 *Let K be any theory of $\mathcal{L}$, and let $\preceq_K$ be any total preorder over $\mathbb{M}$, faithful to K. There exists an extension $\mathcal{L}^+$ of $\mathcal{L}$ containing at most $2^{|\mathcal{P}|} - 3 + |\mathcal{P}|$ extra propositional variables that do not appear in $\mathcal{P}$, at which a restricted Dalal's preorder $\sqsubseteq_{\langle \mathbb{M}^\diamond, K^\diamond \rangle}$ can be defined (where $\mathbb{M}^\diamond$ is a derivative of $\mathbb{M}$ at $\mathcal{L}^+$ and $K^\diamond$ an extension of K in $\mathcal{L}^+$), such that $\sqsubseteq^{\mathcal{L}}_{\langle \mathbb{M}^\diamond, K^\diamond \rangle} = \preceq_K$.*

Proof. This proof relies on the proof of Theorem 3. It suffices to show that the maximum value that n is able to take (in the construction described in the proof of Theorem 3) is $2^{|\mathcal{P}|} - 3 + |\mathcal{P}|$.

By the construction of the worlds in $\mathbb{M}^\diamond$, we derive that *Dist* $(K^\diamond, z_{r_k}) =$ *Dist* $(K^\diamond, z_{r_2}) + |L - L_2| = \max_{r \in L_2} \big(\textit{Dist}\,(K, r)\big) + |L - L_2|$.[13] Given that, for any $r \in \mathbb{M}$, *Dist* $(K, r) \leqslant |\mathcal{P}|$, and the maximum value of $|L - L_2|$ is $2^{|\mathcal{P}|} - 2$ (in the case of a preorder associated with a complete theory K), we have that:

$$\textit{Dist}\,(K^\diamond, z_{r_k}) \leqslant 2^{|\mathcal{P}|} - 2 + |\mathcal{P}|. \quad \textbf{(1)}$$

On the other hand, given that, for any $r_1 \in \mathbb{M}$, the set l_{r_1} contains only positive literals, and, for some $u \in L_k$, the set l_u contains only negative literals, it follows that:

$$\textit{Dist}\,(K^\diamond, z_u) = \textit{Dist}\,(K, u) + n. \quad \textbf{(2)}$$

Combining (1) and (2), we derive that:

$$\textit{Dist}\,(K, u) + n \leqslant 2^{|\mathcal{P}|} - 2 + |\mathcal{P}|. \quad \textbf{(3)}$$

Since, for any $r \notin L_1 = [K]$, *Dist* $(K, r) \geqslant 1$, we derive from (3) that the maximum value that n is able to take is $2^{|\mathcal{P}|} - 3 + |\mathcal{P}|$, as desired. ■

The previous result, essentially, establishes an *upper bound*, in the sense that we do not need more than $2^{|\mathcal{P}|} - 3 + |\mathcal{P}|$ extra propositional variables, in order to construct an extension $\mathcal{L}^+$ of $\mathcal{L}$, at which a restricted Dalal's preorder can be defined, such that its projection to $\mathcal{L}$ is identical to the initial preorder $\preceq_K$.

6.3 Modelling Revision Functions at an Extended Language

In this subsection, we shall present a representation result analogous to Theorem 1 of Katsuno and Mendelzon. First however, let us define the concept of $\mathcal{L}$-to-$\mathcal{L}^+$ *Dalal assignment*.

Definition 12 ($\mathcal{L}$-to-$\mathcal{L}^+$ Dalal Assignment) *Let $\mathcal{L}^+$ be an extension of $\mathcal{L}$. An $\mathcal{L}$-to-$\mathcal{L}^+$ Dalal assignment is a function that maps each theory K of $\mathcal{L}$ to a restricted Dalal's preorder $\sqsubseteq_{\langle \mathbb{M}^\diamond, K^\diamond \rangle}$ over some derivative $\mathbb{M}^\diamond$ of $\mathbb{M}$ at $\mathcal{L}^+$, associated with an extension $K^\diamond$ of K in $\mathcal{L}^+$.*

In a similar way that a faithful assignment specifies a family of preorders $\{\preceq_K\}_{K \in \mathbb{K}}$, an $\mathcal{L}$-to-$\mathcal{L}^+$ Dalal assignment specifies a family of preorders $\{\sqsubseteq_{\langle \mathbb{M}^\diamond, K^\diamond \rangle}\}_{K \in \mathbb{K}}$ (each preorder is defined over some derivative $\mathbb{M}^\diamond$ of $\mathbb{M}$ at $\mathcal{L}^+$, and is associated with an extension $K^\diamond$ of K in $\mathcal{L}^+$).

Theorem 4 *In a sufficiently extended language $\mathcal{L}^+$ of $\mathcal{L}$, any AGM revision function, defined at $\mathcal{L}$, can be constructed via condition (F$*$), by means of an appropriate $\mathcal{L}$-to-$\mathcal{L}^+$ Dalal assignment.*

Proof. The proof follows from the following three facts (see Figure 2):

- **(i)** An $\mathcal{L}$-to-$\mathcal{L}^+$ Dalal assignment assigns to *every* theory K of $\mathcal{L}$ a restricted Dalal's preorder $\sqsubseteq_{\langle \mathbb{M}^\diamond, K^\diamond \rangle}$ over some derivative $\mathbb{M}^\diamond$ of $\mathbb{M}$ at $\mathcal{L}^+$, associated with an extension $K^\diamond$ of K in $\mathcal{L}^+$ (Definition 12).
- **(ii)** The projection of $\sqsubseteq_{\langle \mathbb{M}^\diamond, K^\diamond \rangle}$ to $\mathcal{L}$ constitutes a faithful preorder $\preceq_K$, associated with K (Theorem 3).
- **(iii)** Any AGM revision function $*$, defined at $\mathcal{L}$, is constructed via (F$*$), by means of the family of faithful preorders $\{\preceq_K\}_{K \in \mathbb{K}}$ (Theorem 1). ■

[13] See condition (ii) in the proof of Theorem 3.

$$\underset{\mathcal{L}\text{-to-}\mathcal{L}^{+}\text{ Dalal assignment}}{\left\{ \sqsubseteq_{\langle \mathbb{M}^{\diamond}, K^{\diamond} \rangle} \right\}_{K \in \mathbb{K}}} \xrightarrow{\textit{Projection to } \mathcal{L}} \{\preceq_K\}_{K \in \mathbb{K}} \xleftrightarrow{\quad (\mathbf{F}*) \quad} *$$

Figure 2. An $\mathcal{L}$-to-$\mathcal{L}^+$ Dalal assignment specifies a family of preorders $\left\{ \sqsubseteq_{\langle \mathbb{M}^{\diamond}, K^{\diamond} \rangle} \right\}_{K \in \mathbb{K}}$ (each preorder is defined over some derivative $\mathbb{M}^{\diamond}$ of $\mathbb{M}$ at $\mathcal{L}^+$, and is associated with an extension $K^{\diamond}$ of K in $\mathcal{L}^+$), which, in turn, projected to $\mathcal{L}$, determines a family of preorders $\{\preceq_K\}_{K \in \mathbb{K}}$. The family $\{\preceq_K\}_{K \in \mathbb{K}}$ corresponds, via condition (F$*$), to an AGM revision function $*$, defined at $\mathcal{L}$. Refer to the proof of Theorem 4.

Theorem 4 points out something very interesting: *Any* AGM revision function, defined at $\mathcal{L}$, can be modelled by means of Dalal's rule, defined at a sufficiently extended language. That is to say, the added expressivity of the language results in an augmented capability of modelling revision policies.

Due to well-known computational complexity results [7, 12], the implementation of both *general* and *efficient* belief-revision systems for solving real-world problems has proven to be quite difficult. Against this background, the research focuses mainly to the implementation of concrete "off-the-shelf" revision operators, utilizing a fixed algorithm for encoding the underlying revision policy of an agent, requiring *the least possible* information. One such approach has been, very recently, presented in [10], where an efficient solver for a natural *generalization* of Dalal's operator was developed. As Theorem 4 augments the applicability of the latter, it opens the door to the handling of any type of revision function by systems like that of [10].

Last but not least, from a theoretical viewpoint, proving that any AGM revision function can be modelled by (a modification of) the well-behaved Dalal's operator leads to a *conceptual and ontological reduction*.

7 CONCLUSION

Dalal's revision operator constitutes, undoubtedly, a simple and intuitive construction implementing the belief-revision process. In this article, we proved that — although not in a straightforward way — a sufficient extension of the underlying (propositional) language allows for the modelling of an arbitrary AGM revision function (defined at the initial language), by means of Dalal's approach (defined at the extended language). The enrichment of Dalal's proposal results in a conceptual and ontological reduction, as well as brings the implementation of any type of revision function for practical purposes a step closer.

ACKNOWLEDGEMENTS

This research is co-financed by Greece and the European Union (European Social Fund – ESF) through the Operational Programme "Human Resources Development, Education and Lifelong Learning" in the context of the project "Reinforcement of Postdoctoral Researchers – 2nd Cycle" (MIS-5033021), implemented by the State Scholarships Foundation (IKY).

The authors are grateful to the anonymous reviewers for their fruitful comments on this work.

REFERENCES

[1] Carlos Alchourrón, Peter Gärdenfors, and David Makinson, 'On the logic of theory change: Partial meet contraction and revision functions', *Journal of Symbolic Logic*, **50**(2), 510–530, (1985).

[2] Theofanis Aravanis, Pavlos Peppas, and Mary-Anne Williams, 'Epistemic-entrenchment characterization of Parikh's axiom', in *Proceedings of the 26th International Joint Conference on Artificial Intelligence, IJCAI 2017*, pp. 772–778, Melbourne, Australia, (August 2017).

[3] Theofanis Aravanis, Pavlos Peppas, and Mary-Anne Williams, 'Full characterization of Parikh's relevance-sensitive axiom for belief revision', *Journal of Artificial Intelligence Research*, **66**, 765–792, (2019).

[4] Theofanis Aravanis, Pavlos Peppas, and Mary-Anne Williams, 'An investigation of parametrized difference revision operators', *Annals of Mathematics and Artificial Intelligence*, (2019).

[5] Alexander Borgida, 'Language features for flexible handling of exceptions in information systems', *ACM Transactions on Database Systems*, **10**(4), 563–603, (1985).

[6] Mukesh Dalal, 'Investigations into theory of knowledge base revision: Preliminary report', in *Proceedings of the 7th National Conference of the American Association for Artificial Intelligence, AAAI 1988*, pp. 475–479, (1988).

[7] Thomas Eiter and Georg Gottlob, 'On the complexity of propositional knowledge base revision, updates, and counterfactuals', *Artificial Intelligence*, **57**, 227–270, (1992).

[8] Eduardo Fermé and Sven O. Hansson, *Belief Change: Introduction and Overview*, Springer International Publishing, 2018.

[9] Peter Gärdenfors, *Knowledge in Flux – Modeling the Dynamics of Epistemic States*, MIT Press, Cambridge, Massachusetts, 1988.

[10] Aaron Hunter and John Agapeyev, 'An efficient solver for parametrized difference revision', in *AI 2019: Advances in Artificial Intelligence*, eds., Jixue Liu and James Bailey, volume 11919 of *Lecture Notes in Computer Science*, 143–152, Springer, Cham, (2019).

[11] Hirofumi Katsuno and Alberto Mendelzon, 'Propositional knowledge base revision and minimal change', *Artificial Intelligence*, **52**(3), 263–294, (1991).

[12] Bernhard Nebel, 'How hard is it to revise a belief base?', in *Handbook of Defeasible Reasoning and Uncertainty Management Systems, Vol. 3: Belief Change*, eds., Didier Dubois and Henri Prade, 77–145, Kluwer Academic, (1998).

[13] Maurice Pagnucco and Pavlos Peppas, 'Causality and minimal change demystified', in *Proceedings of the 17th International Joint Conference on Artificial Intelligence, IJCAI 2001*, pp. 125–135, (2001).

[14] Rohit Parikh, 'Beliefs, belief revision, and splitting languages', in *Logic, Language and Computation*, eds., Lawrence S. Moss, Jonathan Ginzburg, and Maarten de Rijke, volume 2, 266–278, CSLI Publications, (1999).

[15] Pavlos Peppas, 'Belief revision', in *Handbook of Knowledge Representation*, eds., Frank van Harmelen, Vladimir Lifschitz, and Bruce Porter, 317–359, Elsevier Science, (2008).

[16] Pavlos Peppas and Mary-Anne Williams, 'Kinetic consistency and relevance in belief revision', in *Proceedings of the 15th European Conference on Logics in Artificial Intelligence, JELIA 2016*, pp. 401–414. Springer International Publishing, (2016).

[17] Pavlos Peppas and Mary-Anne Williams, 'Parametrised difference revision', in *Proceedings of the 16th International Conference on Principles of Knowledge Representation and Reasoning, KR 2018*, pp. 277–286, (2018).

[18] Pavlos Peppas, Mary-Anne Williams, Samir Chopra, and Norman Foo, 'Relevance in belief revision', *Artificial Intelligence*, **229**, 126–138, (December 2015).

[19] Ken Satoh, 'Nonmonotonic reasoning by minimal belief revision', in *Proceedings of the International Conference on Fifth Generation Computer Systems*, pp. 455–462. Springer-Verlag, Tokyo, (1988).

[20] Andreas Weber, 'Updating propositional formulas', in *Proceedings of the 1st Conference on Expert Database Systems*, pp. 487–500, (1986).

[21] Marianne Winslett, 'Reasoning about action using a possible models approach', in *Proceedings of the 7th National Conference of the American Association for Artificial Intelligence, AAAI 1988*, pp. 89–93, (1988).